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Separability and Metrisability in Locally Convex Spaces

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matics.*

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0. Introduction

Summary. It is well-known that a locally convex space E is metrisable if and only if there is a countable local base for its neighbourhoods of 0. It follows that a normed space is separable if and only if its dual unit ball is weak*-metrisable. There is thus a duality between separability and weak*-metrisability; a normed space contains a countable dense subset if and only if the weak* topology on its dual has particularly nice properties. It is not surprising that the original work in Functional Analysis focused on separable Banach spaces.

This thesis is devoted to a study of the relationship between separability and metrisability in the context of locally convex spaces. The duality between separability and weak*-metrisability does not carry over to non-metrisable locally convex spaces; the best that can be said in this case is that the equicontinuous subsets in the dual of a separable locally convex space are weak*-metrisable. To get around this difficulty, we often prefer to use the idea of separability by seminorm: a locally convex space E is separable by seminorm if and only if the equicontinuous subsets of its dual are weak*-metrisable. On any locally convex space E there is a finest topology \mathcal{T}_χ which is coarser than the given topology and which makes E separable by seminorm. A question that arises is under what conditions a space E is \mathcal{T}_χ -complete. In trying to answer this question, we are led to an intriguing binary relation which G.A. Edgar [10] originally defined on the class of Banach spaces. In the first two Chapters of this thesis, we show that many of the results in Edgar's paper can be expressed in terms of the completeness of a space with respect to various topologies. These results touch on such varied topics as the condition of Mazur, weakly compactly gen-

erated Banach spaces and the Pettis Integral Property. It can be argued that even in context of Banach spaces, the use of locally convex topologies throws new light on Edgar's relation. Also in these Chapters we characterise locally convex Grothendieck spaces, and discuss subseries summability and the Orlicz-Pettis Theorem. There is a strong connection between Edgar's relation and some recently-published results on Tauberian operators.

Chapter 3 is devoted to the study of locally convex spaces in which precompact sets are metrisable. Some years ago H. Pfister [29] used the idea of separability by seminorm to show that DF spaces have this property. More recently, B. Cascales and J. Orihuela [4] found an even larger class of spaces of this type. While Cascales and Orihuela appealed to ideas from descriptive set theory, we are able to derive a similar result by more elementary means.

A great deal of work has been done on the relationship between Banach spaces whose separable subspaces have separable duals (the so-called Asplund spaces) and the famous Radon-Nikodým Property. It is easy to show that a Banach space E is an Asplund space if and only if its bounded separable subsets are weakly metrisable. In Chapter 4 we study locally convex spaces in which this is the case. We obtain a result (based on the famous Davis-Figiel-Johnson-Pelczynski construction) that says that under certain conditions, a continuous linear map which maps bounded separable sets onto weakly metrisable subsets factors through an Asplund space. Not only does this generalise a known Banach space result, but its proof is more direct. The second part of Chapter 4 is inspired by certain characterisations of the Radon-Nikodým Property in terms of geometrical properties of bounded sets (so-called "dentability conditions"). We show that if the separable subsets of a given bounded set A in a locally convex space E are weakly metrisable, then the weak*-compact sets in the dual of E satisfy a sort of "dentability condition".

In Chapter 5 we consider holomorphic maps between Banach spaces. Once we have defined a class of Asplund holomorphic maps, we show that a holomorphic map belongs to this class if and only if the same is true of the polynomial maps

which appear in its Taylor expansion about each point. We indicate how results of this type might be put in a unified framework using operator ideals.

Notation and Terminology. Since the notation and terminology used in this thesis are relatively standard, this section is not intended to be a comprehensive list of the terms which will appear. For a general reference on the theory of locally convex spaces, see [19]. Other concepts will be defined when they are needed.

In this work, locally convex spaces are always assumed to be Hausdorff. For any dual pair $\langle E, F \rangle$ of vector spaces, we will use $\sigma(E, F)$, $\beta(E, F)$ and $\mu(E, F)$ to denote the corresponding weak, strong and Mackey topologies on E . The (topological) dual of a locally convex space E is denoted by E' ; the bidual E'' is the space of all $\beta(E', E)$ -continuous linear functionals on E' . It is customary to think of the elements of E as being linear functionals on E' , so that E is identified with the subspace of E'' consisting of the $\sigma(E', E)$ -continuous linear functionals on E' . If $T : E \rightarrow F$ is a continuous linear map between locally convex spaces, then $T' : F' \rightarrow E'$ and $T'' : E'' \rightarrow F''$ denote the first and second adjoints of T . These maps are defined by

$$T'(g) = g \circ T \quad \text{for all } g \in F'$$

and

$$T''(\phi) = \phi \circ T' \quad \text{for all } \phi \in E''$$

respectively. The polar of a subset A of E is

$$A^\circ = \{f \in E' : \sup_{x \in A} |f(x)| \leq 1\}.$$

A subset of E' is *equicontinuous* if and only if it is contained in the polar of a neighbourhood of 0.

Let A be an absolutely convex subset A of a vector space. Associated with A is its linear span, $\text{span } A$. There is a seminorm on $\text{span } A$, given by

$$p_A(x) = \inf\{\lambda > 0 : x \notin \lambda A\} \quad \text{for all } x \in \text{span } A.$$

In some cases, such as when A is a $\sigma(E, E')$ -bounded subset of a locally convex space, p_A is a norm on $\text{span } A$. If this is so, and the linear span of A is complete, then the absolutely convex set A is said to be a *Banach disc*. More generally, we can construct a normed space from $\text{span } A$ by forming the quotient of this space by its subspace

$$\ker p_A = \{x \in \text{span } A : p_A(x) = 0\}.$$

The completion of $\text{span } A / \ker p_A$ is then a Banach space, which we refer to as the *Banach space generated by A* .

A locally convex space E is σ -barrelled if every $\sigma(E', E)$ -bounded sequence in E' is equicontinuous on E . Similarly, E is σ -quasibarrelled if the $\beta(E', E)$ -bounded sequences in E' are equicontinuous on E . If the $\beta(E', E)$ -null sequences in E' are equicontinuous on E , then we say that E is *sequentially quasibarrelled*, and if the $\sigma(E', E)$ -null sequences are equicontinuous, then the locally convex space E is *sequentially barrelled*.

A locally convex space E is *quasinormable* if for each closed absolutely convex neighbourhood U of 0 in E , there is another such neighbourhood, V , with following property: for each $\varepsilon > 0$ there is a bounded subset B of E satisfying $V \subseteq B + \varepsilon U$. If for each U the neighbourhood V can be chosen such that these bounded sets are finite, then E is a *Schwartz space*. Any locally convex space E equipped with its weak topology $\sigma(E, E')$ is a Schwartz space.

A *gDF space* is a locally convex space whose topology can be localised on a fundamental sequence of bounded sets. Every gDF space is both quasinormable and sequentially quasibarrelled. If every $\beta(E', E)$ -bounded countable subset of the dual of a gDF space E is equicontinuous, then E is a *DF space*. When equipped with its strong topology $\beta(F', F)$, the dual of any metrisable locally convex space F is a DF space.

We are going to make frequent use of Grothendieck's Completeness Theorem [19, 9.2.2]. One form of this Theorem states that the completion of a locally convex space E consists of those linear functionals on E' that are $\sigma(E', E)$ -

continuous on the equicontinuous absolutely convex subsets of E' .

Another result which will be used repeatedly is that the closed absolutely convex hull of a precompact metrisable set is always metrisable (see [20, Thm 1.4]). This implies that if A is a bounded $\sigma(E, E')$ -metrisable subset of a locally convex space E , then $A^{\circ\circ} \subseteq E''$ is $\sigma(E'', E')$ -metrisable. (Bounded sets are $\sigma(E'', E')$ -precompact.) Thus the closed absolutely convex hull of any bounded $\sigma(E, E')$ -metrisable set is $\sigma(E, E')$ -metrisable.

1. The associated \mathcal{X} -topology

Separability by seminorm. A locally convex space E is *separable by seminorm* if for each neighbourhood U of 0 in E there is a countable subset C of E with $E = C + U$. This idea appears under various names in the literature: H. Pfister [29] says that such a space is “of countable type”, while L. Drewnowski [8] dubs E “transseparable”. Although these authors have studied the same concept in the context of more general classes of spaces, we shall restrict our attentions to locally convex spaces so that we can use results from duality theory. The terminology “separable by seminorm” can be traced back to [15].

Any separable locally convex space is separable by seminorm. The converse does not hold in general, but is true for metrisable spaces. Separability by seminorm enjoys good permanence properties: it is preserved under the taking of completions, subspaces, products, continuous linear images and projective limits. Every Lindelöf locally convex space is separable by seminorm; Lindelöf spaces bear the same relationship to separability by seminorm as compact sets do to precompactness.

The following is a fundamental characterisation of separability by seminorm; it can be found in [29].

PROPOSITION 1.1. *A locally convex space E is separable by seminorm if and only if the equicontinuous subsets of E' are $\sigma(E', E)$ -metrisable.*

PROOF: Suppose that E is separable by seminorm. To prove that the equicontinuous subsets of E' are $\sigma(E', E)$ -metrisable, it is sufficient to show that the

polar of each neighbourhood U of 0 in E is $\sigma(E', E)$ -metrisable. Let C be a countable subset of E satisfying $E = C + U$. We shall show that $\{F^\circ \cap U^\circ : F \text{ is a finite subset of } C\}$ is a local base for the $\sigma(E', E)$ -neighbourhoods of 0 in U° . First note that the subbasic $\sigma(E', E)$ -neighbourhoods of 0 in U° are of the form $\{x\}^\circ \cap U^\circ$, where $x \in E$. Choose a $y \in C$ and a $z \in U$ such that $2x = y + z$. For each $f \in \{y\}^\circ \cap U^\circ$ we have that

$$|f(x)| \leq \frac{1}{2} |f(y)| + \frac{1}{2} |f(z)| \leq 1,$$

so $\{y\}^\circ \cap U^\circ \subseteq \{x\}^\circ \cap U^\circ$. This means that every $\sigma(E', E)$ -neighbourhood of 0 in U° contains a neighbourhood of the form $F^\circ \cap U^\circ$, where F is a finite subset of C . There are only countably many finite subsets of C , so U° is $\sigma(E', E)$ -metrisable.

Conversely, suppose that the equicontinuous subsets of E' are $\sigma(E', E)$ -metrisable. Given a neighbourhood U of 0 in E , we wish to find a countable subset C of E such that $E = C + U$. First select a closed absolutely convex neighbourhood V such that $2V \subseteq U$. Because V° is $\sigma(E', E)$ -metrisable, there exists a sequence F_1, F_2, F_3, \dots of finite subsets of E such that $\{F_n^\circ \cap V^\circ : n \in \mathbb{N}\}$ is a local base for the $\sigma(E', E)$ -neighbourhoods of 0 in V° . For each $n \in \mathbb{N}$, $F_n^{\circ\circ}$ is a precompact subset of E , so there exists a finite subset G_n of E satisfying $F_n^{\circ\circ} \subseteq G_n + V$. Put $C = \bigcup_{n=1}^{\infty} G_n$. If $x \in E$, then $\{x\}^\circ \cap V^\circ$ is a $\sigma(E', E)$ -neighbourhood of 0 in V° , so $F_n^\circ \cap V^\circ \subseteq \{x\}^\circ$ for some $n \in \mathbb{N}$. Since

$$\begin{aligned} x \in \{x\}^{\circ\circ} &\subseteq (F_n^\circ \cap V^\circ)^\circ \\ &\subseteq (F_n^{\circ\circ} \cup V^{\circ\circ})^{\circ\circ} \\ &\subseteq F_n^{\circ\circ} + V^{\circ\circ} \\ &\subseteq (G_n + V) + V \\ &\subseteq G_n + U \\ &\subseteq C + U, \end{aligned}$$

we have that $E \subseteq C + U$, as required. ■

A locally convex space E that is equipped with its weak topology $\sigma(E, E')$ is always separable by seminorm. This means that unlike ordinary separability,

separability by seminorm is *not* a duality invariant: a locally convex space E that fails to be separable by seminorm when equipped with its given topology is nonetheless separable by seminorm when equipped with its weak topology $\sigma(E, E')$.

From Proposition 1.1 we can see that any Schwartz space is separable by seminorm. It is possible to generalise this observation. A locally convex space E is said to be *semi-weak* (see [3]) if every equicontinuous subset of E' is contained in the $\sigma(E', E)$ -closed absolutely convex hull of an equicontinuous $\sigma(E', E)$ -null sequence. The class of semi-weak locally convex spaces includes the Schwartz spaces. The following Lemma shows that any semi-weak locally convex space is separable by seminorm.

LEMMA 1.2. *Let E be a locally convex space. The $\sigma(E', E)$ -closed absolutely convex hull of any equicontinuous $\sigma(E', E)$ -null sequence in E' is $\sigma(E', E)$ -metrisable.*

PROOF: Any equicontinuous $\sigma(E', E)$ -null sequence in E' is the image of the canonical basis (e_n) in ℓ^1 under a $(\sigma(\ell^1, c_0), \sigma(E', E))$ -continuous linear map. The closed absolutely convex hull of (e_n) in ℓ^1 is $\sigma(\ell^1, c_0)$ -compact and metrisable, so its image under such a map is $\sigma(E', E)$ -metrisable. ■

An important idea which we have used in this proof is that the continuous linear maps from E into c_0 can be identified with the equicontinuous $\sigma(E', E)$ -null sequences in E' . We shall use this idea again shortly.

It is not difficult to see that a locally convex space E is separable by seminorm if and only if every continuous linear map from E into a Banach space F has separable range. Let \mathcal{X} be the operator ideal consisting of those continuous linear maps between Banach spaces that are of this type. (For the theory of operator ideals and related concepts, see [19].) A locally convex topology on a vector space E is an \mathcal{X} -topology if it makes E separable by seminorm. We saw above that $\sigma(E, E')$ is always an \mathcal{X} -topology. The associated \mathcal{X} -topology on a locally convex space E is defined to be the finest \mathcal{X} -topology coarser than the given

topology on E . We see from Proposition 1.1 that the associated \mathcal{X} -topology $\mathcal{T}_{\mathcal{X}}$ on a locally convex space E can be characterised as being the topology of uniform convergence on the $\sigma(E', E)$ -metrisable equicontinuous convex subsets of E' .

The condition of Mazur. In [22], T. Kappeler discusses what he calls “ d -complete” Banach spaces. Recall that a Banach space E satisfies the *condition of Mazur* if every $\sigma(E', E)$ -sequentially continuous linear functional on E' is $\sigma(E', E)$ -continuous. Let $d(E, E')$ be the topology on E of uniform convergence on the $\sigma(E', E)$ -metrisable bounded subsets of E' . Kappeler makes the observation that a Banach space E is $d(E, E')$ -complete if and only if E satisfies the condition of Mazur. Since $d(E, E')$ is simply the associated \mathcal{X} -topology on a Banach space E , we are able to formulate a “locally convex” version of this result.

PROPOSITION 1.3. *If $\mathcal{T}_{\mathcal{X}}$ is the associated \mathcal{X} -topology on a locally convex space E , then the $\mathcal{T}_{\mathcal{X}}$ -completion of E consists of those linear functionals on E' that are $\sigma(E', E)$ -sequentially continuous on the equicontinuous subsets of E' .*

PROOF: According to Grothendieck’s Completeness Theorem [19, 9.2.2], the $\mathcal{T}_{\mathcal{X}}$ -completion of E consists of those linear functionals on E' that are $\sigma(E', E)$ -continuous on the absolutely convex $\sigma(E', E)$ -metrisable equicontinuous subsets of E' . Using Lemma 1.2, we see that these are precisely the linear functionals on E' that are $\sigma(E', E)$ -sequentially continuous on the equicontinuous subsets of E' . ■

It follows that a locally convex space E is complete in its associated \mathcal{X} -topology if and only if every linear functional that is $\sigma(E', E)$ -sequentially continuous on the equicontinuous subsets of E' belongs to E .

A locally convex space E is said to be a *Mazur space* if every sequentially continuous linear functional on E is continuous. The reason for this name is that a Banach space E satisfies the condition of Mazur if and only if E' is a

Mazur space when equipped with its weak* topology $\sigma(E', E)$. A. Wilansky [36] has made a study of Mazur spaces. In his terminology, a Banach space that satisfies the condition of Mazur is a " μB space", while a locally convex space E whose dual forms a Mazur space in its $\sigma(E', E)$ -topology is called a " μlc space". Obviously Wilansky's μlc spaces are closely related to the class of locally convex spaces which are complete in their associated \mathcal{X} -topologies; the distinction between the two classes of spaces results from the inclusion of the word "equicontinuous" in the characterisation of $T_{\mathcal{X}}$.

Since the associated \mathcal{X} -topology $T_{\mathcal{X}}$ on a locally convex space E is coarser than the given topology on E , a $T_{\mathcal{X}}$ -complete locally convex space is always complete in its given topology. (Compare this with [36, 3.1].)

The relation \prec . In [10], G.A. Edgar introduced a binary relation \prec on the class of Banach spaces: he defined $E \prec F$ to mean that for each $\phi \in E'' \setminus E$, there is a continuous linear map $T : E \rightarrow F$ with $T''(\phi) \in F'' \setminus F$. One of the many interesting results in Edgar's paper is that a Banach space E satisfies the condition of Mazur if and only if $E \prec c_0$. Although Edgar noted the possibility of extending the binary relation \prec to the class of locally convex spaces, he did not do so. Since Proposition 1.3 says that the $T_{\mathcal{X}}$ -complete locally convex spaces are a natural generalisation of Banach spaces satisfying the condition of Mazur, it is reasonable to expect that if \prec were extended to locally convex spaces, then it should be possible to prove that a locally convex space E is $T_{\mathcal{X}}$ -complete if and only if $E \prec c_0$. However there is a snag, in that the $T_{\mathcal{X}}$ -completion of a locally convex space E need not be a subspace of E'' . (Consider, for example, an infinite-dimensional normed space with its weak topology.) In order to get around this difficulty, we shall modify Edgar's definition slightly and consider all the linear functionals on E' , not just the ones belonging to E'' .

DEFINITION 1.4. For each pair of locally convex spaces E, F , define $E \prec F$ to mean that each linear functional ϕ on E' satisfies one of the following conditions: either $\phi \in E$, or there exists a continuous linear map $T : E \rightarrow F$ such that $\phi \circ T'$ fails to be $\sigma(F', F)$ -continuous on F' .

It can be seen that we have simply replaced the second adjoint $T'' : E'' \rightarrow F''$ appearing in Edgar's definition by the algebraic adjoint $T'^* : E'^* \rightarrow F'^*$. (Here E'^* and F'^* are the algebraic duals of E' and F' respectively.)

The relation \prec is "functorial", in the sense that if $E \prec F$ then the same relationship holds when the topologies on E and F are replaced by any other pair of locally convex topologies inducing the same continuous linear maps. Thus $E \prec F$ if and only if the same relationship holds when the given topologies on E and F are simultaneously replaced by the weak topologies $\sigma(E, E')$ and $\sigma(F, F')$, or by the associated \mathcal{X} -topologies on E and F . This is an important observation; we shall use it to express some of the results in Edgar's paper in terms of the various locally convex topologies which can be defined on a space E .

Let E and F be Banach spaces. A continuous linear map $T : E \rightarrow F$ is said to be *Tauberian* [21] if $\phi \in E'' \setminus E \implies T''(\phi) \in F'' \setminus F$. The reader will note that there is an obvious similarity between this notion and Edgar's original definition of the binary relation \prec . We shall see that many of the properties of Tauberian operators correspond to properties of \prec . For example, in [21, 3.2] N. Kalton and A. Wilansky have shown that if E and F are Banach spaces, then a continuous linear map $T : E \rightarrow F$ is Tauberian if and only if the following statements are equivalent for each subset A of E :

- (1) A is relatively $\sigma(E, E')$ -compact, and
- (2) A is bounded and $T(A)$ is a relatively $\sigma(F, F')$ -compact subset of F .

(See also [27, Lemma 5].)

PROPOSITION 1.5. *Let E and F be locally convex spaces with $E \prec F$. For each subset A of E , the following statements are equivalent:*

- (1) A is relatively $\sigma(E, E')$ -compact, and
- (2) A is bounded and $T(A)$ is a relatively $\sigma(F, F')$ -compact subset of F for each continuous linear map $T : E \rightarrow F$.

PROOF: The implication (1) \Rightarrow (2) is trivial. Suppose that A satisfies condition

(2). Let B be the $\sigma(E'', E')$ -closure of A in E'' ; because A is bounded, B is $\sigma(E'', E')$ -compact. We wish to show that $B \subseteq E$. Suppose that there exists a $\phi \in B$ that does not belong to E . Since $E \prec F$, there must be a continuous linear map $T : E \rightarrow F$ with $T''(\phi) \notin F$. But because $T(A)$ is relatively $\sigma(F, F')$ -compact, $T''(\phi)$ must lie in the $\sigma(F, F')$ -closure of $T(A)$ in F , a contradiction. ■

In other words if $E \prec F$, then any class of bounded subsets of E whose images under each continuous linear map $T : E \rightarrow F$ are relatively $\sigma(F, F')$ -compact will be relatively $\sigma(E, E')$ -compact in E . Many of the results presented in the next two Chapters rely on this fact in one form or another.

For example, recall that a locally convex space E is *semireflexive* if and only if the bounded absolutely convex subsets of E are relatively $\sigma(E, E')$ -compact. Using the above Proposition, we obtain the following generalisation of [10, Prop. 1].

COROLLARY 1.6. *Let E and F be locally convex spaces with $E \prec F$. If F is semireflexive, then so is E . ■*

Closely related to Proposition 1.5 is the following simple observation.

PROPOSITION 1.7. *Let E and F be locally convex spaces with $E \prec F$. If F is complete, then so is E .*

PROOF: Let ϕ be a linear functional that manages to be $\sigma(E', E)$ -continuous on the equicontinuous absolutely convex subsets of E' . We wish to show that $\phi \in E$. For each continuous linear map $T : E \rightarrow F$, the linear functional $\phi \circ T'$ is $\sigma(F', F)$ -continuous on the equicontinuous absolutely convex subsets of F' . According to Grothendieck's Completeness Theorem, such a functional belongs to F . Since $E \prec F$, this means that $\phi \in E$, as required. ■

The fact that the relation \prec is functorial allows us to deduce a number of results from this Proposition. The next Corollary is representative of these; we shall explore further consequences of Proposition 1.7 in Chapter 2.

Let E and F be Banach spaces, with F satisfying the condition of Mazur. Wilansky has shown that if there exists a Tauberian continuous linear map $T : E \rightarrow F$, then E must also satisfy the condition of Mazur [36, 5.1].

COROLLARY 1.8. *Let E and F be locally convex spaces with $E \prec F$. If F is complete in its associated \mathcal{X} -topology, then so is E . ■*

This Corollary leads to the required analogue of Edgar's result concerning Banach spaces satisfying the condition of Mazur (see [10, Prop. 3]).

PROPOSITION 1.9. *Let $T_{\mathcal{X}}$ be the associated \mathcal{X} -topology on a locally convex space E . Then $E \prec c_0$ if and only if E is $T_{\mathcal{X}}$ -complete.*

PROOF: Since c_0 is a separable Banach space, it is complete in its associated \mathcal{X} -topology. Therefore if $E \prec c_0$, then E is $T_{\mathcal{X}}$ -complete. To obtain the converse, consider a linear functional ϕ on E' that does not belong to E . If E is $T_{\mathcal{X}}$ -complete, then ϕ must fail to be $\sigma(E', E)$ -sequentially continuous on some equicontinuous convex subset of E' . In other words, there must be an equicontinuous $\sigma(E', E)$ -null sequence (f_n) in E' such that $(\phi(f_n)) \notin c_0$. Define $T : E \rightarrow c_0$ by $T(x) = (f_n(x))$; then $\phi \circ T'$ fails to be $\sigma(\ell^1, c_0)$ -continuous on ℓ^1 , as required. ■

2. Applications of Edgar's relation

The Pettis Integral Property. Edgar originally studied Banach spaces satisfying the condition of Mazur because of a connection with Pettis integration. In order to discuss this topic, we shall have to introduce a couple of definitions. (For more details, see [2].) To start with, let (Ω, Σ, μ) be a probability space, and suppose that E is a locally convex space. A map $\Phi : \Omega \rightarrow E$ is bounded if its range $\Phi(\Omega)$ is a bounded subset of E . Such a map is *weakly μ -measurable* if the scalar-valued function $f \circ \Phi$ is μ -measurable for each $f \in E'$. We say that a weakly μ -measurable map $\Phi : \Omega \rightarrow E$ is *μ -Pettis integrable* if for each $S \in \Sigma$, there is an $x \in E$ satisfying

$$f(x) = \int_S f \circ \Phi \, d\mu \quad \text{for all } f \in E'.$$

A locally convex space E has the *μ -Pettis Integral Property* if every bounded weakly measurable map $\Phi : \Omega \rightarrow E$ is μ -Pettis integrable. Edgar has shown (see [10, Prop. 8]) that if E and F are Banach spaces with $E \prec F$, and if F has the μ -Pettis Integral Property for some probability space (Ω, Σ, μ) , then E has the μ -Pettis Integral Property. We can use his proof to obtain a corresponding result for locally convex spaces.

PROPOSITION 2.1. *Let E and F be locally convex spaces with $E \prec F$. If F has the μ -Pettis Integral Property for some probability space (Ω, Σ, μ) , then so does E .*

PROOF: Let $\Phi : \Omega \rightarrow E$ be a bounded weakly μ -measurable map. For each

$S \in \Sigma$, we can define a linear functional ϕ on E' by

$$\phi(f) = \int_S f \circ \Phi \, d\mu \quad \text{for all } f \in E'.$$

Since $E \prec F$, we can prove that $\phi \in E$ by showing that $\phi \circ T' \in F$ for every continuous linear map $T : E \rightarrow F$. If this is the case, it will follow that Φ is μ -Pettis integrable.

Suppose therefore that T is a continuous linear map from E into F . Then $T \circ \Phi$ is a bounded weakly μ -measurable map from Ω into F . Since F has the μ -Pettis Integral Property, $T \circ \Phi$ must be μ -Pettis integrable. Let $y \in F$ be such that

$$g(y) = \int_S g \circ (T \circ \Phi) \, d\mu \quad \text{for all } g \in F'.$$

Then for each $g \in F'$,

$$\begin{aligned} (\phi \circ T')(g) &= \phi(T'(g)) \\ &= \int_S (T'(g)) \circ \Phi \, d\mu \\ &= \int_S (g \circ T) \circ \Phi \, d\mu \\ &= \int_S g \circ (T \circ \Phi) \, d\mu \\ &= g(y). \end{aligned}$$

It follows that $\phi \circ T' = y \in F$. ■

A locally convex space E is said to have the *Pettis Integral Property* if it has the μ -Pettis Integral Property for every probability space (Ω, Σ, μ) . It can be shown that every separable Banach space has the Pettis Integral Property (see [2, pg 301]). Thus in particular c_0 has the Pettis Integral Property. Using Propositions 1.9 and 2.1, we can deduce that whenever a locally convex space E is complete in its associated \mathcal{X} -topology, it has the Pettis Integral Property.

Weakly compactly generated spaces. A Banach space E is said to be *weakly compactly generated (WCG)* if it can be expressed as the closed linear span of one of its $\sigma(E, E')$ -compact subsets. In this section we shall show

that the class of $\mathcal{T}_\mathcal{X}$ -complete locally convex spaces contains not only all weakly compactly generated Banach spaces, but many other spaces besides. Recall that a compact Hausdorff space K is *angelic* (see [12]) if and only if the closure of each subset S of K is the set of limits of sequences in S . Edgar has observed [9, pg 565] that if the closed unit ball of the dual of a Banach space E is $\sigma(E', E)$ -angelic, then E satisfies the condition of Mazur. This result can be re-formulated in terms of locally convex spaces as follows.

PROPOSITION 2.2. *Let E be a complete locally convex space with associated \mathcal{X} -topology $\mathcal{T}_\mathcal{X}$. If each $\sigma(E', E)$ -closed equicontinuous subset of E' is $\sigma(E', E)$ -angelic, then E is $\mathcal{T}_\mathcal{X}$ -complete.*

PROOF: Let M be a $\sigma(E', E)$ -closed equicontinuous subset of E' . Because M is $\sigma(E', E)$ -angelic, any linear functional on E' which is $\sigma(E', E)$ -sequentially continuous on M must be $\sigma(E', E)$ -continuous on M . Since E is complete, it follows that every linear functional on E' which manages to be $\sigma(E', E)$ -sequentially continuous on the $\sigma(E', E)$ -closed equicontinuous subsets of E' is $\sigma(E', E)$ -continuous on E' . The result now follows by Proposition 1.3. ■

Using the Banach space version of this Proposition, Edgar was able to deduce that any closed subspace of a weakly compactly generated Banach space satisfies the condition of Mazur. In order to extend this result to locally convex spaces, we turn to some work of R.J. Hunter and J. Lloyd [18], who have made a study of various locally convex analogues of the class of weakly compactly generated Banach spaces. Amongst other things, they proved the following (see [18, 2.2(b)]).

LEMMA 2.3. *If a locally convex space E is the closed linear span of the union of a sequence of absolutely convex $\sigma(E, E')$ -compact sets, then every $\sigma(E', E)$ -compact subset of E' is $\sigma(E', E)$ -angelic.*

PROOF: A locally convex space E is the closed linear span of the union of a sequence of absolutely convex $\sigma(E, E')$ -compact sets if and only if there is

a metrisable locally convex topology on E' coarser than the Mackey topology $\mu(E', E)$. The result now follows from the “Angelic Lemma” [12, 3.1, 3.10(2)]. ■

This provides us with the following extension of Edgar’s result.

PROPOSITION 2.4. *If a locally convex space E is the closed linear span of the union of a sequence of absolutely convex $\sigma(E, E')$ -compact sets, then every complete subspace of E is complete in its associated \mathcal{X} -topology. ■*

A. Wilansky [36, 3.5] also noted that closed subspaces of weakly compactly generated Banach spaces satisfy the condition of Mazur. He defined a compact Hausdorff space K to be a μ -space if $\mathcal{C}(K)$ (the Banach space of all continuous scalar-valued functions on K) satisfies the condition of Mazur. Recall that a topological space K is said to be *Eberlein-compact* if it is $\sigma(E, E')$ -homeomorphic to a $\sigma(E, E')$ -compact subset of a Banach space E . It is known that a compact Hausdorff space K is Eberlein-compact if and only if $\mathcal{C}(K)$ is a weakly compactly generated Banach space. Using this fact, Wilansky was able to deduce that every Eberlein-compact topological space is a μ -space (see [36, 4.1]). He asked whether the converse might be true. In fact this conjecture can be answered in the negative: M. Talagrand [34] has shown that there exist compact Hausdorff spaces K for which the equicontinuous subsets of $\mathcal{C}(K)'$ are $\sigma(\mathcal{C}(K)', \mathcal{C}(K))$ -angelic, but which fail to be Eberlein-compact. Such a space is an example of a μ -space that is not Eberlein-compact.

In passing, we note that Wilansky gives an example (assuming the Continuum Hypothesis) of a Banach space which satisfies the condition of Mazur, but which is not a subspace of any weakly compactly generated Banach space [36, 3.7] (see also [6, pg 226]). This provides us with an example of a locally convex space which is complete in its associated \mathcal{X} -topology, but which is not a subspace of a locally convex space of the type mentioned in the previous Proposition.

Grothendieck spaces. In his paper [36], Wilansky also discussed the subject of Grothendieck Banach spaces (he called them “GB spaces”). As he put it,

these spaces “lie at the opposite end of a spectrum” from the Banach spaces satisfying the condition of Mazur. It turns out that Grothendieck Banach spaces are interesting in their own right: we shall see that they are related to compact operators in much the same way as Banach spaces satisfying the condition of Mazur are related to Tauberian operators. However, just as we have preferred to work with \mathcal{T}_χ -complete locally convex spaces rather than with μB spaces, we shall discard Wilansky’s definition of a GB space in favour of one which seems to be better suited to the locally convex situation. In [13], F.J. Freniche has defined a locally convex space E to be a *Grothendieck space* if the $\sigma(E', E)$ -convergent equicontinuous sequences in E' are $\sigma(E', E'')$ -convergent. (Wilansky’s version omits the word “equicontinuous”.) The following result is easy to establish.

PROPOSITION 2.5. *For a locally convex space E , the following statements are equivalent:*

- (1) E is a Grothendieck space,
- (2) every continuous linear map T from E into a \mathcal{T}_χ -complete locally convex space F satisfies $T''(E'') \subseteq F$, and
- (3) every continuous linear map T from E into c_0 satisfies $T''(E'') \subseteq c_0$. ■

If a locally convex space E is quasinormable, then a continuous linear map T from E into a Banach space F is weakly compact if and only if $T''(E'') \subseteq F$ (see [19, 17.2.7]). We can therefore derive the following version of [11, 3.1].

PROPOSITION 2.6. *If a locally convex space E is quasinormable, then the following statements are equivalent:*

- (1) E is a Grothendieck space,
- (2) every continuous linear map T from E into a weakly compactly generated Banach space F is weakly compact,
- (3) every continuous linear map T from E into a separable Banach space F is weakly compact, and
- (4) every continuous linear map T from E into c_0 is weakly compact. ■

Condition (4) should be compared with the following result of M. Lindström’s:

A quasinormable locally convex space E is a Schwartz space if and only if every continuous linear map from E into c_0 is compact (see [23, pg 426]).

The topology $\gamma(E, E')$. We have already seen that it is possible to extend many of Edgar's results regarding his binary relation \prec to locally convex spaces. As promised earlier, we shall now explore some other consequences of Proposition 1.7. Most of these are based on the fact that the relation \prec is functorial. It should become apparent that even in the Banach space setting, expressing Edgar's results in terms of locally convex topologies provides a useful basis for the study of the relation \prec .

Our first result is a generalisation of [10, Prop. 6]. For any locally convex space E , let $\gamma(E, E')$ be the topology on E of uniform convergence on the $\sigma(E', E)$ -bounded sequences in E' . Note that a locally convex space E is σ -barrelled if and only if $\gamma(E, E')$ is coarser than the given topology on E .

PROPOSITION 2.7. *Let E be a σ -barrelled locally convex space. Then $E \prec \ell^\infty$ if and only if E is $\gamma(E, E')$ -complete.*

PROOF: From Proposition 1.7 and the fact that the relation \prec is functorial, we get that if $E \prec F$ and F is $\gamma(F, F')$ -complete, then E is $\gamma(E, E')$ -complete. Since ℓ^∞ is $\gamma(\ell^\infty, \ell^{\infty'})$ -complete, one half of the equivalence follows.

To see the converse, note that the continuous linear maps from E into ℓ^∞ can be identified with the equicontinuous sequences in E' . If E is σ -barrelled, then these sequences are the $\sigma(E', E)$ -bounded sequences in E' . Suppose that ϕ is a linear functional on E' that does not belong to E . Since E is $\gamma(E, E')$ -complete, ϕ must fail to be $\sigma(E', E)$ -continuous on the $\sigma(E', E)$ -closed absolutely convex hull of a $\sigma(E', E)$ -bounded sequence (f_n) in E' . Let $T : E \rightarrow \ell^\infty$ be defined by $T(x) = (f_n(x))$ for all $x \in E$. Then $\phi \circ T'$ is not $\sigma(\ell^{\infty'}, \ell^\infty)$ -continuous on $\ell^{\infty'}$, as required. ■

If we replace $\gamma(E, E')$ in this Proposition with the topology on E of uniform convergence on the equicontinuous $\sigma(E', E)$ -bounded sequences in E' , then we

can drop the condition that E be σ -barrelled. Conversely, if to Proposition 1.9 we add the requirement that the locally convex space E be sequentially barrelled, then we can re-phrase that Proposition to say that $E \prec F$ if and only if E is $d(E, E')$ -complete. In other words, provided that the locally convex space E satisfies a suitably strong barrelledness condition, we can omit the word “equicontinuous” from the definition of the topology we place on E .

Geitz’s condition. Once again let (Ω, Σ, μ) be a probability space. If E is a Banach space, then every bounded weakly μ -measurable function $\Phi : \Omega \rightarrow E$ gives rise to a μ -absolutely continuous vector measure $m_\Phi : \Sigma \rightarrow E''$. This measure is known as the *indefinite Dunford integral* of Φ , and is defined for each $S \in \Sigma$ by

$$f(m_\Phi(S)) = \int_S f \circ \Phi \, d\mu \quad \text{for all } f \in E'.$$

By the definition of μ -Pettis integrability, the range of m_Φ lies in E exactly when Φ is μ -Pettis integrable. At the start of this Chapter we used the relation \prec to determine when this is true for all bounded weakly μ -measurable functions $\Phi : \Omega \rightarrow E$. We shall now extend that work slightly, by allowing E to be a locally convex space and by considering other vector measures. We shall be concerned with the following question: *Under what conditions is the range of every μ -absolutely continuous vector measure $\nu : \Sigma \rightarrow E''$ contained in E ?*

Let $\overline{\text{co}} A$ denote the $\sigma(E'', E')$ -closed convex hull of a subset A of E'' . For each $S \in \Sigma$, put

$$\Sigma_S^+ = \{A \in \Sigma : A \subseteq S, \mu(A) > 0\}.$$

The *average range* of a vector measure $\nu : \Sigma \rightarrow E''$ on a measurable set $S \in \Sigma$ is defined to be

$$\text{AR}_S(\nu) = \left\{ \frac{\nu(A)}{\mu(A)} : A \in \Sigma_S^+ \right\}.$$

In [16], R.F. Geitz used this concept to characterise the Pettis integrable functions $\Phi : \Omega \rightarrow E$, where E is a Banach space. It follows from the Hahn–Banach Theorem that in this case

$$\overline{\text{co}} \text{AR}_S(m_\Phi) = \bigcap \{ \overline{\text{co}} \Phi(S \setminus B) : B \in \Sigma, \mu(B) = 0 \}$$

(see [16, 2.2] and [33]). The intersection of this set with E is known as the *core of Φ relative to S* (see [2, pg 311]). It follows from Geitz's work that if the function $\Phi : \Omega \rightarrow E$ has non-empty core relative to each $S \in \Sigma_\Omega^+$, then Φ is μ -Pettis integrable (cf. [2, Thm 7.4.16(d)]).

Following Freniche [14], we shall say that a vector measure $\nu : \Sigma \rightarrow E''$ satisfies *Geitz's condition* if $E \cap \overline{\text{co}} \text{AR}_S(\nu) \neq \emptyset$ for all $S \in \Sigma_\Omega^+$. It can be seen that if E is a Banach space, then the indefinite Dunford integral of a bounded measurable function $\Phi : \Omega \rightarrow E$ satisfies Geitz's condition if and only if Φ has non-empty core relative to each $S \in \Sigma_\Omega^+$. We shall therefore say that *Geitz's condition suffices in E* (for the probability space (Ω, Σ, μ)) if the range of every μ -absolutely continuous vector measure $\nu : \Sigma \rightarrow E''$ satisfying Geitz's condition is contained in E .

PROPOSITION 2.8. *Let E and F be locally convex spaces with $E \prec F$. If Geitz's condition suffices in F for the probability space (Ω, Σ, μ) , then it suffices in E .*

PROOF: Let $\nu : \Sigma \rightarrow E''$ be a μ -absolutely continuous vector measure satisfying Geitz's condition. Since $E \prec F$, shall show that $T''(\nu(S)) \in F$ for each $S \in \Sigma$ and for each continuous linear map $T : E \rightarrow F$.

For each T , the map $\nu_T : \Sigma \rightarrow F''$ given by $\nu_T : S \mapsto T''(\nu(S))$ is a μ -absolutely continuous vector measure. For each $S \in \Sigma$, we have that

$$\text{AR}_S(\nu_T) = \left\{ \frac{T''(\nu(A))}{\mu(A)} : A \in \Sigma_S^+ \right\} = T''(\text{AR}_S(\nu)).$$

Now because ν satisfies Geitz's condition,

$$\begin{aligned} E \cap \overline{\text{co}} \text{AR}(\nu) \neq \emptyset &\implies F \cap T''(\overline{\text{co}} \text{AR}_S(\nu)) \neq \emptyset \\ &\implies F \cap \overline{\text{co}} \text{AR}_S(\nu_T) \neq \emptyset \end{aligned}$$

for all $S \in \Sigma_\Omega^+$. This means that the vector measure ν_T satisfies Geitz's condition, and so $T''(\nu(S)) = \nu_T(S) \in F$ for all $S \in \Sigma$, as required. ■

Freniche has shown [14, Thm 1] that if E is a Banach space and $\nu : \Sigma \rightarrow E''$ is a vector measure satisfying Geitz's condition, then for each $S \in \Sigma$, $\nu(S)$ is a

$\sigma(E', E)$ -sequentially continuous linear functional on E' . As he points out, this implies that Geitz's condition suffices in every Banach space satisfying the condition of Mazur. Proposition 2.8 allows us to extend this result to a locally convex space E which is complete in its associated \mathcal{K} -topology. In particular, we see that Geitz's condition suffices in a locally convex space E if the $\sigma(E', E)$ -closed equicontinuous subsets of E' are $\sigma(E', E)$ -angelic. This observation generalises another result in [14], which says that Geitz's condition suffices in any Banach space E whose dual unit ball is $\sigma(E', E)$ -sequentially compact.

Freniche has also shown [14, Thm 3] that if E is a Banach space, then any μ -absolutely continuous vector measure $\nu : \Sigma \rightarrow E''$ which satisfies Geitz's condition and whose range is not contained in E gives rise to a μ -absolutely continuous vector measure $\hat{\nu} : \Sigma \rightarrow \ell^\infty$ which satisfies Geitz's condition and whose range is not contained in ℓ^∞ . From this and Proposition 2.7, we can deduce that if Geitz's condition suffices in ℓ^∞ for a probability space (Ω, Σ, μ) , then the same is true in every $\gamma(E, E')$ -complete σ -barrelled space E .

Property (X) and the Orlicz–Pettis Theorem. G. Godefroy and M. Talagrand [17] define a Banach space E to have *property (X)* if E contains every linear functional $\phi \in E''$ which satisfies

$$\phi\left(\sum_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} \phi(f_n)$$

for each sequence (f_n) in E' obeying

$$\sum_{n=1}^{\infty} |f_n(x)| < \infty \quad \text{for all } x \in E.$$

Edgar [10, Prop. 10] has proved that a Banach space E has property (X) if and only if $E \prec \ell^1$.

The sequences (f_n) appearing in the above definition are known as $\sigma(E', E)$ -unconditionally Cauchy sequences. In order to extend Edgar's result to locally convex spaces, we shall make use of a generalisation of these sequences. In

doing so, we shall see that “property (X)” is related to the famous Orlicz–Pettis Theorem (see [19, 14.6.4] and [6, pg 25]).

A sequence (x_n) in a locally convex space E is said to be *subseries summable* if for each subsequence (x_{n_k}) of (x_n) , the series $\sum_{k=1}^{\infty} x_{n_k}$ converges in E . It can be shown that a sequence (x_n) in E is subseries summable if and only if the set of partial sums

$$\left\{ \sum_{n \in N} x_n : N \text{ is a finite subset of } \mathbb{N} \right\}$$

is a relatively $\sigma(E, E')$ -compact subset of E (see [19, 14.6.4] and [25]). It follows that subseries summability is a duality invariant: a sequence is subseries summable with respect to the given topology on E if and only if it is $\sigma(E, E')$ -subseries summable. This last observation is of course a generalisation of the classic Orlicz–Pettis Theorem, which states that a sequence in a Banach space E is norm-subseries summable if and only if it is $\sigma(E, E')$ -subseries summable.

For any locally convex space E , let E^s consist of those linear functionals f on E that satisfy

$$f\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} f(x_n)$$

for each subseries summable sequence (x_n) in E . Note that $E' \subseteq E^s$; we call E^s the *Orlicz–Pettis dual* of E (cf. [35]).

Let m_0 be the space of those scalar-valued sequences that take on only finitely many values. According to Schur’s Lemma (see [19, 10.5.2]), the $\sigma(\ell^1, m_0)$ -compact sets are exactly the norm-compact subsets of ℓ^1 .

PROPOSITION 2.9. *For a sequence (x_n) in a locally convex space E , the following statements are equivalent:*

- (1) (x_n) is subseries summable in E ,
- (2) for each $(\mu_n) \in m_0$, the series $\sum_{n=1}^{\infty} \mu_n x_n$ converges in E ,
- (3) for each $(\xi_n) \in \ell^\infty$, the series $\sum_{n=1}^{\infty} \xi_n x_n$ is $\mu(E, E^s)$ -convergent, and
- (4) (x_n) is $\mu(E, E^s)$ -subseries summable.

PROOF: (1) \Rightarrow (2): Suppose that $(\mu_n) \in m_0$ takes on the values a_1, a_2, \dots, a_m .

For each i , put

$$A_i = \{n \in \mathbb{N} : \mu_n = a_i\}.$$

We then have that $\mathbb{N} = A_1 \cup A_2 \cup \dots \cup A_m$. Without any loss of generality, we can assume that each of these sets A_i is infinite, so that we can arrange its elements into a strictly increasing sequence (n_{ik}) . Since $\mu_{n_{ik}} = a_i$ for all $k \in \mathbb{N}$,

$$\sum_{k=1}^{\infty} \mu_{n_{ik}} x_{n_{ik}} = a_i \sum_{k=1}^{\infty} x_{n_{ik}}.$$

(The series on the right converges in E because of the subseries summability of (x_n) .) By considering partial sums, we see that the series $\sum_{n=1}^{\infty} \mu_n x_n$ converges in E , with

$$\sum_{n=1}^{\infty} \mu_n x_n = \sum_{i=1}^m \sum_{k=1}^{\infty} a_i x_{n_{ik}}.$$

(2) \Rightarrow (3): First note that if $\sum_{n=1}^{\infty} \mu_n x_n$ converges in E for every $(\mu_n) \in m_0$, then for each such (μ_n) the sequence $(\mu_n x_n)$ is subseries summable in E . By the definition of E^s , it follows that

$$f\left(\sum_{n=1}^{\infty} \mu_n x_n\right) = \sum_{n=1}^{\infty} \mu_n f(x_n) \quad \text{for all } f \in E^s.$$

Thus the series $\sum_{n=1}^{\infty} \mu_n x_n$ is $\sigma(E, E^s)$ -convergent. By considering sequences (μ_n) that take on the values 0 and 1, we see that (x_n) is $\sigma(E, E^s)$ -subseries summable in E . Thus for each $f \in E^s$, $\sum_{n=1}^{\infty} f(x_n)$ is an unconditionally convergent series of scalars. Such a series is always absolutely convergent, which means that $(f(x_n)) \in \ell^1$. We can therefore define a map $T : E^s \rightarrow \ell^1$ by

$$T(f) = (f(x_n)) \quad \text{for all } f \in E^s.$$

This map has an adjoint $T' : m_0 \rightarrow E$ given by

$$T'((\mu_n)) = \sum_{n=1}^{\infty} \mu_n x_n \quad \text{for all } (\mu_n) \in m_0.$$

It follows that $T : E^s \rightarrow \ell^1$ is $(\sigma(E^s, E), \sigma(\ell^1, m_0))$ -continuous, and therefore maps $\sigma(E^s, E)$ -compact subsets of E^s onto $\sigma(\ell^1, m_0)$ -compact subsets of ℓ^1 . According to Schur's Theorem, the $\sigma(\ell^1, m_0)$ -compact subsets of ℓ^1 are norm-compact, so this means that T maps the absolutely convex $\mu(E, E^s)$ -equicontinuous subsets of E^s onto relatively norm-compact subsets of ℓ^1 . Taking adjoints, we find that T' maps the absolutely convex $\mu(\ell^1, \ell^\infty)$ -equicontinuous subsets of m_0 onto relatively $\mu(E, E^s)$ -compact subsets of E . The sections of each sequence $(\xi_n) \in \ell^\infty$ form an $\mu(\ell^1, \ell^\infty)$ -equicontinuous subset of m_0 ; the image of this set under T' is the set of all partial sums of $\sum_{n=1}^\infty \xi_n x_n$. Since we have shown that the latter set is relatively $\mu(E, E^s)$ -compact in E , the series $\sum_{n=1}^\infty \xi_n x_n$ is $\mu(E, E^s)$ -convergent.

(3) \Rightarrow (4): Given a subsequence (x_{n_k}) of (x_n) , define $(\xi_m) \in \ell^\infty$ by

$$\xi_m = \begin{cases} 1 & \text{if } m = n_k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\sum_{k=1}^\infty x_{n_k} = \sum_{n=1}^\infty \xi_n x_n$; the latter series $\mu(E, E^s)$ -converges in E .

(4) \Rightarrow (1): This follows immediately from the fact that $\mu(E, E^s)$ is finer than the given topology on E . ■

The topology on a locally convex space E is said to be an *Orlicz-Pettis topology* if there is no strictly finer locally convex topology on E with the same subseries summable sequences (see [5]). Using the above Proposition, we see that the topology \mathcal{T} on E is an Orlicz-Pettis topology if and only if it equals $\mu(E, E^s)$. In other words, \mathcal{T} is an Orlicz-Pettis topology if and only if it is the Mackey topology on E and every linear functional in E^s belongs to E' (cf. [35]).

PROPOSITION 2.10. *Let E and F be locally convex spaces with $E \prec F$. If $\mu(F', F)$ is an Orlicz-Pettis topology, then so is $\mu(E', E)$.*

PROOF: We have to show that when E' is equipped with its weak* topology $\sigma(E', E)$, its Orlicz-Pettis dual is E . Suppose that ϕ is a linear functional on

E' , satisfying

$$\phi\left(\sum_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} \phi(f_n)$$

for each $\sigma(E', E)$ -subseries summable sequence (f_n) in E' . Since $E \prec F$, we can show that $\phi \in E$ by proving that $\phi \circ T' \in F$ for each continuous linear map $T : E \rightarrow F$. Let (g_n) be a $\sigma(F', F)$ -subseries summable sequence in F' . For each continuous linear map $T : E \rightarrow F$, the sequence $(T'(g_n))$ is $\sigma(E', E)$ -subseries summable, so

$$\begin{aligned} \phi \circ T'\left(\sum_{n=1}^{\infty} g_n\right) &= \phi\left(\sum_{n=1}^{\infty} T'(g_n)\right) \\ &= \sum_{n=1}^{\infty} \phi \circ T'(g_n) \end{aligned}$$

Since $\mu(F', F)$ is an Orlicz-Pettis topology, it follows that $\phi \circ T' \in F$, as required. ■

We can now formulate a generalisation of Edgar's result [10, Prop. 10] regarding Banach spaces with property (X).

PROPOSITION 2.11. *Let E be a locally convex space with the property that every $\sigma(E', E)$ -subseries summable sequence in E' is equicontinuous on E . Then $\mu(E', E)$ is an Orlicz-Pettis topology if and only if $E \prec \ell^1$.*

PROOF: One half of this equivalence follows from the fact that $\mu(\ell^\infty, \ell^1)$ is an Orlicz-Pettis topology; this is a consequence of the proof of Proposition 2.9. The converse hinges on the fact that for any locally convex space E , the continuous linear maps $T : E \rightarrow \ell^1$ are in one-to-one correspondence with the equicontinuous $\sigma(E', E)$ -subseries summable sequences in E' . ■

COROLLARY 2.12. *Let E be a locally convex space with the property that every $\sigma(E', E)$ -subseries summable sequence in E' is equicontinuous on E . If $\mu(E', E)$ is an Orlicz-Pettis topology, then E is complete in its associated \mathcal{X} -topology. ■*

Let E be an infinite-dimensional Banach space with property (X). According to the above Proposition, $\mu(E', E)$ is an Orlicz-Pettis topology. However when

we equip the same space E with its weak topology, we obtain an example of a locally convex space for which $\mu(E', E)$ is an Orlicz-Pettis topology, but which is not itself complete in its associated \mathcal{X} -topology. This illustrates why it is necessary to specify that the $\sigma(E', E)$ -subseries summable sequences in E' be equicontinuous on E .

Bornological spaces and locally-null sequences. The relation \prec can be used to formulate a variety of similar results involving the Mackey topology $\mu(E', E)$. A sequence (x_n) in a locally convex space E is *locally-null* if it converges to 0 in the normed space generated by a bounded absolutely convex subset B of E . Let $\nu(E', E)$ denote the topology on E' of uniform convergence on the locally-null sequences in E . It can be shown (see [19, 13.2.4]) that the topology on a locally convex space E is bornological if and only if it is the Mackey topology $\mu(E, E')$ on E and E' is $\nu(E', E)$ -complete.

PROPOSITION 2.13. *Let E and F be locally convex spaces with $E \prec F$. If $\mu(F', F)$ is a bornological topology, then so is $\mu(E', E)$.*

PROOF: Since the image of a locally-null sequence under a continuous linear map is again locally-null, every continuous linear map $T : E \rightarrow F$ is $(\nu(E, E'), \nu(F, F'))$ -continuous. The result thus follows by Proposition 1.7 and the fact that the relation \prec is functorial. ■

If the bounded set B used in the definition of a locally-null sequence is a Banach disc (so that the normed space generated by B is complete), then the sequence (x_n) is said to be *fast-null*. Just as before, the topology on a locally convex space E is ultrabornological if and only if it equals the Mackey topology $\mu(E, E')$ on E and E' is complete with respect to the topology of uniform convergence on the fast-null sequences in E .

PROPOSITION 2.14. *Let E and F be locally convex spaces with $E \prec F$. If $\mu(F', F)$ is a ultrabornological topology, then so is $\mu(E', E)$. ■*

It is easy to modify the proof of Proposition 1.7 to show that if $E \prec F$ and F is sequentially complete, then E is sequentially complete. Again appealing to the fact that \prec is functorial, we obtain the following extension of [10, Prop. 9]:

PROPOSITION 2.15. *Let E and F be locally convex spaces with $E \prec F$. If F is $\sigma(F, F')$ -sequentially complete, then E is $\sigma(E, E')$ -sequentially complete. ■*

3. The metrisability of precompact sets

Polar duals and the topology of precompact convergence. Let E'_p denote the *polar dual* of a locally convex space E . This is E' equipped with the topology of uniform convergence on the precompact subsets of E . Note that the equicontinuous subsets of E' are precompact in E'_p .

PROPOSITION 3.1. *If the polar dual E'_p of a locally convex space E is separable by seminorm, then the precompact subsets of E are metrisable.*

PROOF: On the precompact subsets of E , $\sigma(E'', E')$ coincides with the given topology on E . These subsets are equicontinuous on E'_p , so they are $\sigma(E'', E')$ -metrisable. ■

The associated \mathcal{X} -topology on E'_p is the topology of uniform convergence on the metrisable precompact subsets of E . Thus E'_p is separable by seminorm precisely when the precompact subsets of E are metrisable. We can extend this result to spaces of continuous functions (see also [32, IV.1.2]). Recall that a topological space X is *Tychonoff* if it is both Hausdorff and completely regular.

PROPOSITION 3.2. *Let $\mathcal{C}(X)$ denote the space of all continuous scalar-valued functions on a Tychonoff space X . Then $\mathcal{C}(X)$, equipped with the compact-open topology, is separable by seminorm if and only if the compact subsets of X are metrisable.*

PROOF: It is sufficient to show that every continuous linear map from $\mathcal{C}(X)$

into a Banach space F has a separable range. For each compact subset K of X , define a map $J_K : \mathcal{C}(X) \rightarrow \mathcal{C}(K)$ by $J_K(f) = f|_K$ for all $f \in \mathcal{C}(X)$. This map is surjective since each $g \in \mathcal{C}(K)$ can be extended to an $f \in \mathcal{C}(X)$ (see [19, 1.7.1]). Since every continuous linear map from $\mathcal{C}(X)$ into a Banach space F factors through one of these maps J_K , the result now follows from the fact that a compact space K is metrisable if and only if $\mathcal{C}(K)$ is separable (see [19, 3.10.3]). ■

Baire space and the metrisability of precompact sets. Let \mathcal{N} be the set of all functions $\alpha : \mathbb{N} \rightarrow \mathbb{N}$. For each $\alpha, \beta \in \mathcal{N}$, define $\alpha \leq \beta$ to mean that $\alpha(k) \leq \beta(k)$ for all $k \in \mathbb{N}$. If $\alpha \in \mathcal{N}$ and $n \in \mathbb{N}$, put

$$\alpha|n = \{\beta \in \mathcal{N} : \alpha(k) = \beta(k) \text{ for } k = 1, 2, \dots, n\}.$$

Note that $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$, so \mathcal{N} is equipped with a natural product topology. With this topology \mathcal{N} is a separable completely metrisable space*. The basic neighbourhoods of each element α of \mathcal{N} are the sets $\alpha|n$. The space \mathcal{N} is of great importance in descriptive set theory — it is sometimes referred to as *Baire space* and turns out to be homeomorphic to the space of irrational numbers. We shall not explore the topological properties of \mathcal{N} at this stage, as we only need it as an indexing set.

The elements of \mathcal{N} can of course be thought of as being sequences of positive integers. It is sometimes convenient to identify $\alpha|n$ with the finite sequence $\alpha(1), \alpha(2), \dots, \alpha(n)$. Note that each $\alpha|n$ can be written in the form

$$\alpha|n = \bigcup \{\beta|n+1 : \beta \in \alpha|n\}.$$

An important point is that the union on the right-hand side only involves countably many distinct sets, which can be indexed by the values of $\beta(n+1)$, where β ranges through $\alpha|n$.

In [4] B. Cascales and J. Orihuela study the class \mathcal{G} of all locally convex spaces

*Such a space is sometimes called a *Polish space*.

E whose duals contain a family $\{A_\alpha : \alpha \in \mathcal{N}\}$ of subsets satisfying the following conditions:

- (1) $\bigcup_{\alpha \in \mathcal{N}} A_\alpha = E'$,
- (2) $(\forall \alpha, \beta \in \mathcal{N}) \alpha \leq \beta \Rightarrow A_\alpha \subseteq A_\beta$, and
- (3) the countable subsets of each A_α are equicontinuous on E .

This class \mathcal{G} includes both the DF and the LF spaces, and enjoys a wide range of permanence properties. Using ideas from descriptive set theory, Cascales and Orihuela have shown that the precompact subsets of each $E \in \mathcal{G}$ are metrisable. Their basic approach is to show that if K is a compact subset of the completion of such a locally convex space E , then $\mathcal{C}(K)$ admits what they call a “ \mathcal{K} -analytic structure”, and is therefore Lindelöf. We would like to derive the same result by more elementary methods, without recourse to descriptive set theory.

THEOREM 3.3. *Let E be a locally convex space. If there exists a family $\{P(\alpha) : \alpha \in \mathcal{N}\}$ of precompact subsets of E such that*

- (1) $\bigcup_{\alpha \in \mathcal{N}} P(\alpha) = E$, and
- (2) $(\forall \alpha, \beta \in \mathcal{N}) \alpha \leq \beta \Rightarrow P(\alpha) \subseteq P(\beta)$,

then E is separable by seminorm.

PROOF: Suppose that E is not separable by seminorm. Using Zorn’s Lemma, we can find a neighbourhood U of 0 in E and an uncountable subset A of E such that $(x - y) \in U \Rightarrow x = y$ for all $x, y \in A$. No infinite subset of A can be precompact, yet we shall select a sequence of distinct points from A so that it lies in one of the $P(\alpha)$ ’s. This violates the precompactness of that $P(\alpha)$.

For each $\alpha \in \mathcal{N}$ and $n \in \mathbb{N}$, put

$$P[\alpha|n] = \bigcup \{P(\beta) : \beta \in \alpha|n\}.$$

First note that $A \subseteq \bigcup \{P[\alpha|1] : \alpha \in \mathcal{N}\}$. This union comprises only countably many distinct sets, so we can apply the Pigeon-hole Principle to choose an $\alpha_1 \in \mathcal{N}$ such that $P[\alpha_1|1]$ contains uncountably many points from A . In fact, we can choose α_1 so that $P(\alpha_1)$ itself contains at least one point x_1 from A .

Suppose that for some $n \in \mathbb{N}$, we have selected $\alpha_n \in \mathcal{N}$ and $x_1, x_2, \dots, x_n \in A$, and $P[\alpha_n|n]$ contains uncountably many points from A . Note that

$$P[\alpha_n|n] = \bigcup \{P[\beta|n+1] : \beta \in \alpha_n|n\}.$$

Again this union involves only countably many distinct sets. There must be an $\alpha_{n+1} \in \alpha_n|n$ such that $P[\alpha_{n+1}|n+1]$ contains uncountably many points from A . With a little care we can choose this α_{n+1} so that $P(\alpha_{n+1})$ contains an x_{n+1} that belongs to A and which is different from x_1, x_2, \dots, x_n .

For each $k \in \mathbb{N}$, we have that $\alpha_n(k) = \alpha_k(k)$ for all $n > k$. Define $\gamma \in \mathcal{N}$ by

$$\gamma(k) = \max\{\alpha_1(k), \alpha_2(k), \dots, \alpha_k(k)\} \quad \text{for all } k \in \mathbb{N}.$$

Note that each $\alpha_n \leq \gamma$. This means that $x_n \in P(\alpha_n) \subseteq P(\gamma)$ for all $n \in \mathbb{N}$, leading to the desired contradiction. ■

The statement of the first corollary to this Theorem appears as Satz 1 in [29].

COROLLARY 3.4. *Let E be a locally convex space. If there exists a surjective linear map T from a metrisable locally convex space F onto E , such that the image under T of each bounded sequence in F is a precompact subset of E , then E is separable by seminorm.*

PROOF: Let $\{V_n : n \in \mathbb{N}\}$ be a countable local base for the neighbourhoods of 0 in F consisting of absolutely convex sets. For each $\alpha \in \mathcal{N}$, $\bigcap_{k=1}^{\infty} \alpha(k)V_k$ is a bounded subset of F , so we can define $P(\alpha) = T(\bigcap_{k=1}^{\infty} \alpha(k)V_k)$. ■

This Corollary is usually applied in the case when T is the identity map on a space F equipped with two locally convex topologies \mathcal{T}_1 and \mathcal{T}_2 , where $\mathcal{T}_1 \subseteq \mathcal{T}_2$, \mathcal{T}_2 is metrisable and the \mathcal{T}_2 -bounded subsets of F are \mathcal{T}_1 -precompact. For example, if a locally convex space E contains a fundamental sequence of bounded sets, then its dual E' is $\beta(E', E)$ -metrisable. If E is in addition σ -quasibarrelled, then every $\beta(E', E)$ -bounded sequence in E' is equicontinuous on E , and therefore forms a precompact subset of E'_p . It follows from the above

result that E'_p is separable by seminorm, and so the precompact subsets of E are metrisable. Note in particular that every DF space satisfies these two conditions, so we are able to deduce (as Pfister did) that the precompact subsets of a DF space are metrisable.

The condition that E be σ -quasibarrelled cannot be weakened to E being sequentially quasibarrelled. To see this, let F be a non-separable reflexive Banach space, and suppose that E is the same space equipped with the topology of uniform convergence on the $\beta(F', F)$ -precompact subsets of F' . Because E is a gDF space, it is sequentially quasibarrelled [19, 12.5.6]. The unit ball K of F is precompact in E , but it cannot be separable in E because it is not separable in F . This means that K is a precompact set in E that fails to be metrisable.

Note that a subset K of a locally convex space E is precompact if and only if every countable subset of K is precompact. We can therefore apply Proposition 3.1 and Theorem 3.3 to obtain the following version of Cascales and Orihuela's result.

COROLLARY 3.5. *Let E be a locally convex space. If there exists a family $\{P(\alpha) : \alpha \in \mathcal{N}\}$ of precompact subsets of E'_p such that*

- (1) $\bigcup_{\alpha \in \mathcal{N}} P(\alpha) = E'$, and
- (2) $(\forall \alpha, \beta \in \mathcal{N}) \alpha \leq \beta \Rightarrow P(\alpha) \subseteq P(\beta)$,

then the precompact subsets of E are metrisable. ■

4. Asplund spaces

The $\sigma(E, E')$ -metrisability of bounded separable sets. A Banach space E is said to be an *Asplund space* if every continuous convex real-valued function defined on a non-empty open convex subset D of E is Fréchet-differentiable at each point of a dense G_δ -subset of D . We shall not use this definition directly; for our purposes it will be more convenient to use the fact that a Banach space E is an Asplund space if and only if each separable subspace of E has a separable dual (see [30, 2.34]). As can be seen from their definition, Asplund spaces are of interest in convex analysis and differentiation theory; what makes them particularly important is that they are also related to the Radon-Nikodým Property. For a detailed exposition of the theory of Asplund spaces, and in particular their relationship to vector measures, the reader is referred to the book by R. Bourgin [2]. Recently R. Phelps has published an introduction [30] to the geometrical aspects of the theory.

It can be shown (see [2, 5.4.3]) that a Banach space E is an Asplund space if and only if every bounded separable subset of E is $\sigma(E, E')$ -metrisable. It is thus possible to generalise the class of Asplund Banach spaces by considering the class of locally convex spaces E whose bounded separable subsets are $\sigma(E, E')$ -metrisable. The following simple observation provides a link between the study of such spaces and the ideas considered in the previous Chapters. Given a locally convex space E , we shall define $\gamma(E', E)$ to be the topology on E' of uniform convergence on the bounded sequences in E .

PROPOSITION 4.1. *The bounded separable subsets of a locally convex space E*

are $\sigma(E, E')$ -metrisable if and only if $\gamma(E', E)$ is an \mathcal{X} -topology.

PROOF: Suppose first that the bounded separable subsets of E are $\sigma(E, E')$ -metrisable. Each $\gamma(E', E)$ -equicontinuous subset A of E'' is contained in the bipolar of an absolutely convex bounded separable subset B of E . This set B is $\sigma(E'', E')$ -metrisable, hence so is A . We thus have that every $\gamma(E', E)$ -equicontinuous subset of E'' is $\sigma(E'', E')$ -metrisable, so from Proposition 1.1 we can deduce that $\gamma(E', E)$ is an \mathcal{X} -topology.

Conversely, if $\gamma(E', E)$ is an \mathcal{X} -topology, then every bounded separable subset of E is $\gamma(E', E)$ -equicontinuous, and is therefore $\sigma(E', E'')$ -metrisable. ■

Factoring Asplund operators. We can generalise the class of locally convex spaces mentioned above even further by studying continuous linear maps $T : E \rightarrow F$ between locally convex spaces that map the bounded separable subsets of the domain space E onto $\sigma(F, F')$ -metrisable subsets of the range space F . Our aim in this Section is to establish a factorisation result for such maps similar to the Davis-Figiel-Johnson-Pelczynski Theorem for weakly compact operators (cf. [6, pg 227], [19, 17.2.10]). First we shall prove two technical results which are of independent interest.

LEMMA 4.2. *Let F be a metrisable locally convex space. Suppose that (W_k) and (N_k) are sequences of absolutely convex subsets of F satisfying*

- (1) $\{W_k : k \in \mathbb{N}\}$ is a local base for the neighbourhoods of 0 in F ,
- (2) each N_k is bounded in F , and
- (3) $2W_{k+1} \subseteq W_k$ and $2N_k \subseteq N_{k+1}$ for all $k \in \mathbb{N}$.

Put $W_0 = F$ and define

$$A = \sum_{k=1}^{\infty} (N_k \cap W_{k-1}) \text{ and } B = \bigcap_{k=1}^{\infty} (N_k + W_k).$$

Then A and B are bounded subsets of F satisfying $A \subseteq 2B \subseteq 3A$. Moreover, the closure of B satisfies $\overline{B} \subseteq 2B$.

PROOF: For each $n \in \mathbb{N}$,

$$A \subseteq \sum_{k=1}^n N_k + \sum_{k=n+1}^{\infty} W_{k-1} \subseteq \sum_{k=0}^{n-1} \frac{1}{2^k} N_n + \sum_{k=0}^{\infty} \frac{1}{2^k} W_n \subseteq 2N_n + 2W_n.$$

From this, it follows that A is bounded in F , and also that

$$A \subseteq 2 \bigcap_{n=1}^{\infty} (N_n + W_n) = 2B.$$

To see that $B \subseteq \frac{3}{2}A$, consider any $x \in B$. For each $k \in \mathbb{N}$, there must be a $w_k \in W_k$ and an $z_k \in N_k$ such that $x = w_k + z_k$. Put $y_k = z_{k+1} - z_k$; then

$$y_k = (x - w_{k+1}) - (x - w_k) = w_k - w_{k+1}.$$

We have on the one hand that

$$z_{k+1} - z_k \in N_{k+1} + N_k \subseteq \frac{3}{2}N_{k+1}$$

while on the other

$$w_k - w_{k+1} \in W_k + W_{k+1} \subseteq \frac{3}{2}W_k.$$

We conclude that $y_k \in \frac{3}{2}(N_{k+1} \cap W_k)$ for all $k \in \mathbb{N}$. Since

$$\sum_{k=1}^n y_k = z_{n+1} - z_1 = (x - w_{n+1}) - z_1 \in (x - z_1) + W_{n+1}$$

for each $n \in \mathbb{N}$, the series $\sum_{k=1}^{\infty} y_k$ must converge in F . In fact, $\sum_{k=1}^{\infty} y_k = x - z_1$, so

$$x = z_1 + \sum_{k=1}^{\infty} y_k \in N_1 + \sum_{k=1}^{\infty} \frac{3}{2}(N_{k+1} \cap W_k) \subseteq \frac{3}{2}A.$$

Thus $B \subseteq \frac{3}{2}A$.

Finally, because $\overline{B} = \bigcap_{n=1}^{\infty} (B + W_n)$, we have that

$$\overline{B} \subseteq B + W_n \subseteq (N_k + W_k) + W_n \quad \text{for all } n, k \in \mathbb{N}.$$

In particular, this means that

$$\overline{B} \subseteq N_n + W_n + W_n \subseteq 2(N_n + W_n) \quad \text{for all } n \in \mathbb{N}.$$

Thus

$$\overline{B} \subseteq 2 \bigcap_{n=1}^{\infty} (N_n + W_n) = 2B. \quad \blacksquare$$

The next result is reminiscent of what J. Diestel calls "Grothendieck's Lemma" (see [6, pg 227]). It says that if a subset C of a normed space F can be uniformly approximated by a sequence of $\sigma(F, F')$ -metrizable bounded sets, then C is itself $\sigma(F, F')$ -metrizable.

LEMMA 4.3. *Let W be the unit ball of a normed space F and suppose that C is a subset of F . If for each $m \in \mathbb{N}$, there is a bounded $\sigma(F, F')$ -metrizable subset C_m of F satisfying*

$$C \subseteq C_m + 2^{-m}W,$$

then C is $\sigma(F, F')$ -metrizable.

PROOF: Put $S = \{g \in F' : \|g\| = 1\}$. To show that C is $\sigma(F, F')$ -metrizable, it is sufficient to find a countable subset D of S such that for each $g \in S$ and $\varepsilon > 0$, there is an $f \in D$ satisfying

$$\sup_{y \in C} |f(y) - g(y)| < \varepsilon.$$

For each $m \in \mathbb{N}$, there is a countable subset D_m of S with the following property: for each $g \in S$ and $\varepsilon > 0$, there is an $f \in D_m$ satisfying

$$\sup_{x \in C_m} |f(x) - g(x)| < \frac{1}{3}\varepsilon.$$

Put $D = \bigcup_{m=1}^{\infty} D_m$. Given $g \in S$ and $\varepsilon > 0$, first choose $m \in \mathbb{N}$ so that $2^{-m} < \frac{1}{3}\varepsilon$, and then select $f \in D_m$ so that the above inequality is satisfied.

Now consider any $y \in C$. Since $C \subseteq C_m + 2^{-m}W$, there is an $x \in C_m$ such that $\|x - y\| \leq 2^{-m}$. Thus

$$\begin{aligned} |f(y) - g(y)| &= |f(x) - g(x) - f(x - y) + g(x - y)| \\ &\leq |f(x) - g(x)| + \|f\| \cdot \|x - y\| + \|g\| \cdot \|x - y\| \\ &< \frac{1}{3}\varepsilon + 2^{-m} + 2^{-m} \\ &< \varepsilon, \end{aligned}$$

as required. ■

THEOREM 4.4. *Let E be a quasinormable locally convex space, and suppose that F is a Banach space. For each continuous linear map $T : E \rightarrow F$, the following statements are equivalent:*

- (1) *T maps each bounded sequence in E into an absolutely convex $\sigma(F, F')$ -metrisable subset of F , and*
- (2) *there exists an Asplund space Z together with continuous linear maps $R : E \rightarrow Z$ and $S : Z \rightarrow F$ such that $T = S \circ R$.*

PROOF: (1) \Rightarrow (2): Put $U = \{x \in E : \|T(x)\| \leq 1\}$. By the quasinormability of E , there exists a neighbourhood V of 0 in E such that for each $k \in \mathbb{N}$ there is a bounded subset M_k of E satisfying

$$V \subseteq M_k + 2^{-k}U.$$

Without loss of generality, we can assume that each M_k is absolutely convex and that $2M_k \subseteq M_{k+1}$ for all k .

Put $W_0 = F$ and then for each $k \in \mathbb{N}$, define $W_k = \{y \in F : \|y\| \leq 2^{-k}\}$ and $N_k = T(M_k)$. Put $A = \sum_{k=1}^{\infty} (N_k \cap W_{k-1})$ and $B = \bigcap_{k=1}^{\infty} (N_k + W_k)$. Let Z be the normed space generated by A . According to Lemma 4.2, $A \subseteq 2B \subseteq 3A$ and $\overline{B} \subseteq 2B$, so the bounded sets A , B and \overline{B} generate equivalent norms on Z . Since \overline{B} is a closed subset of the complete space F , we can see that Z is a Banach space. Because $T(V) \subseteq B$, the linear map $T : E \rightarrow F$ factors through Z . If $S : Z \rightarrow F$ is the inclusion map, we can define $R : E \rightarrow Z$ by $T = S \circ R$.

We need to prove that Z is an Asplund space. To do this, we shall first show that the weak topologies $\sigma(Z, Z')$ and $\sigma(F, F')$ coincide on the bounded subsets of Z . It then suffices to prove that every sequence in A is contained in an absolutely convex $\sigma(F, F')$ -metrizable subset of F .

For each $m \in \mathbb{N}$, let $X_m = (F, \|\cdot\|_m)$ be the normed space generated by $\bigcap_{k=1}^m (N_k + W_k)$. Because $\bigcap_{k=1}^m (N_k + W_k)$ is a bounded subset of F containing W_m , each of these spaces X_m is isomorphic to F .

Let $c((X_m))$ be the space of sequences (x_m) in F with the property that $\lim_{m \rightarrow \infty} \|x_m\|_m$ exists. This space can be equipped with the supremum norm

$$\|(x_m)\| = \sup_{m \in \mathbb{N}} \|x_m\|_m.$$

Note that if $x \in B$, then

$$\|x\|_m \leq \|x\|_{m+1} \leq 1$$

for all $m \in \mathbb{N}$, and so the constant sequence (x, x, x, \dots) belongs to $c((X_m))$. Since B generates an equivalent norm on Z , it follows that Z is isomorphic to the subspace X of $c((X_m))$ consisting of all such sequences. We can therefore identify $\sigma(Z, Z')$ with the weak topology $\sigma(X, X')$.

The dual of $c((X_m))$ is the Cartesian ℓ^1 -product space $\ell^1((X'_m))$ (the space of sequences (f_m) in F' satisfying $\sum_{m=1}^{\infty} \|f_m\|_m < \infty$). The weak topology $\sigma(X, X')$ is therefore the restriction of $\sigma(c((X_m)), \ell^1((X'_m)))$ to X . Let $\varphi((X'_m))$ be the space of sequences in F' with only finitely many non-zero terms. Because $\varphi((X'_m))$ is norm-dense in the space $\ell^1((X'_m))$, the two topologies $\sigma(c((X_m)), \ell^1((X'_m)))$ and $\sigma(c((X_m)), \varphi((X'_m)))$ coincide on the bounded subsets of $c((X_m))$.

Because Z is isomorphic to X , the restriction of $\sigma(c((X_m)), \varphi((X'_m)))$ to X can be identified with the restriction of $\sigma(F, F')$ to Z : in fact, if $(x, x, x, \dots) \in X$ and $(f_1, f_2, \dots, f_n, 0, 0, \dots) \in \varphi(X'_m)$, then

$$\langle (x, x, x, \dots), (f_1, f_2, \dots, f_n, 0, 0, \dots) \rangle = f_1(x) + f_2(x) + \dots + f_n(x),$$

where $x \in Z$ and $f_1, f_2, \dots, f_n \in F'$. We can therefore conclude that $\sigma(Z, Z')$ and $\sigma(F, F')$ coincide on the bounded subsets of Z .

We shall now show that every sequence (y_n) in A is contained in an absolutely convex $\sigma(F, F')$ -metrisable subset of F . Since $A = \sum_{k=1}^{\infty} (N_k \cap W_{k-1})$, for each $n \in \mathbb{N}$ there is a sequence (x_{nk}) in E satisfying $y_n = \sum_{k=1}^{\infty} T(x_{nk})$ and with $x_{nk} \in M_k \cap 2^{-k+1}U$ for all $k \in \mathbb{N}$. For each $m \in \mathbb{N}$, the sequence $(\sum_{k=1}^m x_{nk})$ lies in the bounded set $\sum_{k=1}^m M_k$, so its image under T is contained in an absolutely convex bounded $\sigma(F, F')$ -metrisable subset C_m of A . Clearly $y_n \in C_m + W_{m-1}$ for all $m, n \in \mathbb{N}$. We can therefore apply Lemma 4.3 to deduce that the closed absolutely convex hull C of $\{y_n : n \in \mathbb{N}\}$ is $\sigma(F, F')$ -metrisable.

(2) \Rightarrow (1): If (x_n) is a bounded sequence in E , then $\{R(x_n) : n \in \mathbb{N}\}$ is a bounded separable subset of the Asplund space Z . Such a subset is $\sigma(Z, Z')$ -precompact and metrisable, so its image under S is $\sigma(F, F')$ -metrisable (see [20, Lemma 1.2]). ■

A number of people have studied continuous linear maps between Banach spaces that factor through Asplund spaces (see [2, Section 5.3]). Theorem 4.4 not only allows this work to be extended to maps defined on more general quasinormable spaces, it gives a simple characterisation of these maps in terms of their behaviour on sequences.

Dentable sets in locally convex spaces. Much of the interest in Asplund spaces is motivated by a very deep result: a Banach space E is an Asplund space if and only if its dual E' has the Radon–Nikodým Property. The Radon–Nikodým Property can be defined in many ways; again we refer to [2] for a detailed account of the ramifications of this important property. A number of the characterisations of the Radon–Nikodým Property can be described as being geometrical in nature: they involve some form of what is known as the “dentability” of the bounded sets in E' . A variety of theorems characterising the Radon–Nikodým Property in a dual Banach space E' therefore take the following form: a subset S of E' has the Radon–Nikodým Property if and only

if the bounded subsets of S satisfy some or other dentability condition. In this section we will adopt a somewhat different approach, one which is in a sense dual to that just described. Our aim will be to answer the following question: *If the separable subsets of a bounded set A are $\sigma(E, E')$ -metrisable, then what can be said about the bounded subsets of E' ?*

As before, we shall try to work as much as possible in the context of a general locally convex space E . Since E might not be a barrelled space, it makes sense to look at the $\sigma(E', E)$ -compact subsets of E' as the appropriate generalisations of the bounded subsets of the dual. If A and B are non-empty subsets of E and E' respectively, then we define

$$\text{diam}_A B = \sup\{|f(x) - g(x)| : x \in A, f, g \in B\}$$

(provided this supremum exists). In other words, $\text{diam}_A B$ is the diameter of B with respect to the seminorm on E' generated by A° . Now let us define $A \vdash B$ to mean that for each $\varepsilon > 0$, there is a non-empty relatively $\sigma(E', E)$ -open subset W of B with $\text{diam}_A W < \varepsilon$.

Suppose that C is a non-empty $\sigma(E', E)$ -bounded subset of E' . For each $y \in E$, define

$$M(C, y) = \sup_{f \in C} \text{Re } f(y).$$

(Here $\text{Re } f(y)$ denotes the real part of $f(y)$.) The slice of C determined by $y \in E$ and $\alpha > 0$ is

$$S(C, y, \alpha) = \{f \in C : \text{Re } f(y) > M(C, y) - \alpha\}.$$

We define $A \vdash_S C$ to mean that for each $\varepsilon > 0$ there is a $y \in E$ and an $\alpha > 0$ satisfying $\text{diam}_A S(C, y, \alpha) < \varepsilon$.

Each $S(C, y, \alpha)$ is a non-empty relatively $\sigma(E', E)$ -open subset of C . Thus

$$A \vdash_S C \implies A \vdash C.$$

LEMMA 4.5. *Let A be a non-empty subset of a locally convex space E . Suppose that C is the $\sigma(E', E)$ -closed convex hull of a non-empty subset B of E' . If $\text{diam}_A B < \varepsilon$ for some $\varepsilon > 0$, then $\text{diam}_A C \leq 2\varepsilon$.*

PROOF: If $\text{diam}_A B < \varepsilon$, then $B \subseteq f + \varepsilon A^\circ$ for any $f \in B$. Since $f + \varepsilon A^\circ$ is a $\sigma(E', E)$ -closed convex set containing B , it follows that $C \subseteq f + \varepsilon A^\circ$. Thus $\text{diam}_A C \leq 2\varepsilon$. ■

If C is the $\sigma(E', E)$ -closed convex hull of a non-empty $\sigma(E', E)$ -bounded subset B of E' , then we can use the Hahn-Banach Theorem to show that

$$A \vdash_S B \iff A \vdash_S C.$$

Let $\text{ext } C$ denote the set of extreme points of C .

PROPOSITION 4.6. *Let A be a non-empty subset of a locally convex space E . Suppose that C is a non-empty $\sigma(E', E)$ -compact convex subset of E' . If B is the $\sigma(E', E)$ -closure of $\text{ext } C$, then*

$$A \vdash B \implies A \vdash_S C.$$

PROOF: Suppose that $A \vdash B$. Given an $\varepsilon > 0$, we must find a $y \in E$ and an $\alpha > 0$ such that $\text{diam}_A S(C, y, \alpha) < \varepsilon$.

Because $A \vdash B$, we are able to choose a $\sigma(E', E)$ -open set V such that $B \cap V \neq \emptyset$ and $\text{diam}_A(B \cap V) < \frac{1}{3}\varepsilon$. If $B \subseteq V$, then $\text{diam}_A B < \frac{1}{3}\varepsilon$, and so by the preceding Lemma $\text{diam}_A C \leq \frac{2}{3}\varepsilon < \varepsilon$. If this is the case, then $\text{diam}_A S(C, y, \alpha) < \varepsilon$ for any $y \in E$ and $\alpha > 0$.

We now have to consider the case when $B \not\subseteq V$. Because $\text{ext } C$ is $\sigma(E', E)$ -dense in B , the set V must contain an extreme point f_0 of C . Let C_1 be the $\sigma(E', E)$ -closed convex hull of $C \setminus V$. According to Milman's Converse to the Krein-Milman Theorem [6, pg 151], $C \setminus V$ contains all the extreme points of C_1 . It does not contain f_0 , which is an extreme point of C , so $C_1 \neq C$. We can therefore apply the Hahn-Banach Theorem to find a $y \in E$ satisfying

$$M(C_1, y) < \text{Re } f_0(y) \leq M(C, y).$$

Choose $\alpha > 0$ so that

$$M(C_1, y) < M(C, y) - \alpha < \text{Re } f_0(y).$$

We then have that $f_0 \in S(C, y, \alpha)$, but $C_1 \cap S(C, y, \alpha) = \emptyset$. Thus $S(C, y, \alpha) \subseteq C \cap V$. Since $C \cap V$ is contained in the $\sigma(E', E)$ -closed convex hull of $B \cap V$ and $\text{diam}_A(B \cap V) < \frac{1}{3}\varepsilon$, Lemma 4.5 says that

$$\text{diam}_A S(C, y, \alpha) \leq \text{diam}_A(C \cap V) \leq \frac{2}{3}\varepsilon < \varepsilon. \blacksquare$$

COROLLARY 4.7. *If A is a non-empty subset of a locally convex space E , then the following statements are equivalent:*

- (1) $A \vdash B$ for each non-empty $\sigma(E', E)$ -compact subset B of E' , and
- (2) $A \vdash_S C$ for each non-empty $\sigma(E', E)$ -compact convex subset C of E' . \blacksquare

We are now in a position to relate these ideas to the $\sigma(E, E')$ -metrisability of the separable subsets of A . The proof of the following Theorem is based on [2, pg 94], which in turn is based on [28].

THEOREM 4.8. *Let A be a non-empty bounded subset of a locally convex space E . If all the separable subsets of A are $\sigma(E, E')$ -metrisable, then $A \vdash B$ for each non-empty $\sigma(E', E)$ -compact subset B of E' .*

PROOF: Suppose there is a non-empty $\sigma(E', E)$ -compact set B that does not satisfy $A \vdash B$. This means there exists an $\varepsilon > 0$ such that $\text{diam}_A W \geq \varepsilon$ for each non-empty relatively $\sigma(E', E)$ -open subset W of B . We shall construct a sequence (x_k) in A together with an uncountable subset H of B that will satisfy the following condition: for each pair f, g of distinct elements of H , there is an $k \in \mathbb{N}$ satisfying

$$|f(x_k) - g(x_k)| > \frac{\varepsilon}{4}.$$

The closed absolutely convex hull of $\{x_k : k \in \mathbb{N}\}$ is then clearly not $\sigma(E, E')$ -metrisable, contrary to the hypothesis that all the separable subsets of A are $\sigma(E, E')$ -metrisable.

We shall construct the sequence (x_k) by induction on k . At the same time, we will generate a sequence W_1, W_2, W_3, \dots of non-empty relatively $\sigma(E', E)$ -open subsets of B that we will use to construct the uncountable set H .

Start by putting $W_1 = B$. For each $k \in \mathbb{N}$ we proceed as follows.

Notice that because of our choice of B , $\text{diam}_A W_k \geq \varepsilon$. We are thus able to choose an $x_k \in A$ and elements g_{2k}, g_{2k+1} of W_k such that

$$|g_{2k}(x_k) - g_{2k+1}(x_k)| > \frac{3\varepsilon}{4}.$$

Now define

$$W_{2k} = \{f \in W_k : |f(x_k) - g_{2k}(x_k)| < \frac{\varepsilon}{4}\}$$

and

$$W_{2k+1} = \{f \in W_k : |f(x_k) - g_{2k+1}(x_k)| < \frac{\varepsilon}{4}\}$$

before moving on to the next value of k .

Once we have generated the sequence (x_k) and the sets W_1, W_2, W_3, \dots , it is possible to start constructing the set H . For each $k \in \mathbb{N}$, let B_k be the $\sigma(E', E)$ -closure of W_k . Note that B_k is a non-empty $\sigma(E', E)$ -compact subset of B . The family of sets $\{B_k : k \in \mathbb{N}\}$ exhibits a "tree-like" structure: for each $k \in \mathbb{N}$, B_{2k} and B_{2k+1} are disjoint non-empty subsets of B_k . In fact, the triangle inequality shows that

$$|f(x_k) - g(x_k)| > \frac{\varepsilon}{4}$$

for all $f \in B_{2k}$ and $g \in B_{2k+1}$.

Let \mathcal{I} be the collection of all functions $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ satisfying the following conditions: $\alpha(1) = 1$, and for each $i \in \mathbb{N}$, either $\alpha(i+1) = 2\alpha(i)$ or $\alpha(i+1) = 2\alpha(i)+1$. There are uncountably many such functions. We can use the elements of \mathcal{I} to index the "branches" of the tree $\{B_k : k \in \mathbb{N}\}$. For each $\alpha \in \mathcal{I}$, the sets $B_{\alpha(1)}, B_{\alpha(2)}, B_{\alpha(3)}, \dots$ form a decreasing sequence of non-empty $\sigma(E', E)$ -compact subsets of B . Any such sequence has a non-empty intersection, so for each $\alpha \in \mathcal{I}$ we can select an $h_\alpha \in \bigcap_{i=1}^{\infty} B_{\alpha(i)}$.

We claim that $H = \{h_\alpha : \alpha \in \mathcal{I}\}$ is the desired uncountable subset of B . To see this, let α and β be distinct elements of \mathcal{I} . Choose $i \in \mathbb{N}$ so that $\alpha(i) = \beta(i)$, but $\alpha(i+1) \neq \beta(i+1)$. Put $k = \alpha(i)$; then either $h_\alpha \in B_{2k}$ and $h_\beta \in B_{2k+1}$, or

$h_\beta \in B_{2k}$ and $h_\alpha \in B_{2k+1}$. In both cases it follows that

$$|h_\alpha(x_k) - h_\beta(x_k)| > \frac{\varepsilon}{4},$$

as required. ■

5. Asplund holomorphic maps

Infinite-dimensional holomorphy. The classical theory of complex-valued functions of a single complex variable can be generalised to vector-valued functions of vector variables. It turns out that there is a natural way of defining a space of “polynomial maps” between two Banach spaces E and F . It is thus possible to define a continuous map $f : E \rightarrow F$ to be *holomorphic* if it possesses a suitable “Taylor expansion” about each point $a \in E$. (We shall give a more formal definition below.) A holomorphic map $f : E \rightarrow F$ is said to be *compact* if f maps a neighbourhood of each $a \in E$ onto a relatively compact subset of F . R.M. Aron and M. Schottenloher have shown (see [1]) that f is compact if and only if the polynomial maps appearing in the Taylor expansions of f about each $a \in E$ are compact, and that this is the case if and only if f exhibits this behaviour at 0. R.A. Ryan [31] later showed that the same statements hold if the word “compact” is replaced throughout by “weakly compact”. Ryan was also able to extend the Davis–Figiel–Johnson–Pelczynski Theorem for weakly compact linear maps: a k -homogeneous polynomial map $p : E \rightarrow F$ is *weakly compact* if and only if p factors through some reflexive Banach space Z .

In a recent paper [24], M. Lindström has announced similar results for another class of holomorphic maps. This time the word “compact” gets replaced by the phrase “conditionally weakly compact”. (A subset B of a Banach space F is said to be *conditionally weakly compact* if every sequence in B has a $\sigma(F, F')$ -Cauchy subsequence.) Lindström has been able to show that every k -homogeneous polynomial map which belongs to this class can be factorised through a Banach space which does not contain a copy of ℓ^1 . Because there is

a connection here with Rosenthal's celebrated ℓ^1 -Theorem, it is natural to call these maps *Rosenthal maps*.

The main goal of this Chapter is to extend our Theorem 4.4 in the same way. Before doing so, we require a few definitions. A comprehensive reference on infinite-dimensional holomorphy is the volume by S. Dineen [7]; a more concise account is contained in the book by L. Nachbin [26]. Most of the results in this section can be found in either of these references.

Throughout this Chapter, E and F will be a fixed pair of complex Banach spaces*. For each $k \in \mathbb{N}$, let $\mathcal{L}(^k E; F)$ denote the space of all continuous k -linear maps from E^k into F . Let $\Delta_k : E \rightarrow E^k$ be the diagonal map given by $\Delta_k : x \mapsto (x, x, x, \dots, x)$. A continuous map $p : E \rightarrow F$ is said to be a k -homogeneous polynomial map if it is of the form $p = h \circ \Delta_k$ for some $h \in \mathcal{L}(^k E; F)$. Put

$$\mathcal{P}(^k E; F) = \{h \circ \Delta_k : h \in \mathcal{L}(^k E; F)\} \quad \text{for all } k \in \mathbb{N}.$$

We can extend this definition to the case $k = 0$ by letting $\mathcal{P}(^0 E; F)$ be the space of all constant maps from E into F . Each $\mathcal{P}(^k E; F)$ is then a linear subspace of the space of all continuous maps from E into F ; define $\mathcal{P}(E; F) = \bigoplus_{k=0}^{\infty} \mathcal{P}(^k E; F)$. The elements of $\mathcal{P}(E; F)$ are the (continuous) polynomial maps from E into F .

For each $k \in \mathbb{N}$, let $\tilde{\otimes}_{\pi}^k E$ denote the k -fold completed projective tensor product

$$E \tilde{\otimes}_{\pi} E \tilde{\otimes}_{\pi} \dots \tilde{\otimes}_{\pi} E.$$

There is a continuous k -linear map $\chi_k : E^k \rightarrow \tilde{\otimes}_{\pi}^k E$ given by

$$\chi_k(x_1, x_2, \dots, x_k) = x_1 \otimes x_2 \otimes \dots \otimes x_k \quad \text{for all } (x_1, x_2, \dots, x_k) \in E^k.$$

This map has an important canonical property: to each $h \in \mathcal{L}(^k E; F)$ there corresponds a unique continuous linear map $\bar{h} : \tilde{\otimes}_{\pi}^k E \rightarrow F$ for which $\bar{h} \circ \chi_k =$

*Little is gained by extending the results to locally convex spaces, and in order to discuss holomorphic maps it is essential to use complex scalars.

h . This means that $\mathcal{L}({}^k E; F)$ is isomorphic to the space $\mathcal{L}(\tilde{\otimes}_\pi^k E; F)$ of all continuous linear maps from $\tilde{\otimes}_\pi^k E$ into F . Note that the closed unit ball of $\tilde{\otimes}_\pi^k E$ is the closed absolutely convex hull of $\chi_k(U^k)$, where U is the closed unit ball of E .

A k -linear map $h : E^k \rightarrow F$ is said to be *symmetric* if for each permutation σ of the integers $1, 2, \dots, k$, it satisfies

$$h(x_1, x_2, \dots, x_k) = h(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)})$$

for all $(x_1, x_2, \dots, x_k) \in E^k$.

The following result is fundamental; its proof may be found in [7].

THE POLARISATION FORMULA. *If $h \in \mathcal{L}({}^k E; F)$ is symmetric, then*

$$h(x_1, x_2, \dots, x_k) = \frac{1}{2^k k!} \sum_{\epsilon_j = \pm 1} \epsilon_1 \epsilon_2 \dots \epsilon_k h \circ \Delta_k(\epsilon_1 x_1 + \epsilon_2 x_2 + \dots + \epsilon_k x_k)$$

for all $(x_1, x_2, \dots, x_k) \in E^k$. ■

One consequence of this Formula is that for each k -homogeneous polynomial map $p : E \rightarrow F$, there is exactly one symmetric continuous k -linear map $h : E^k \rightarrow F$ for which $p = h \circ \Delta_k$. We call the corresponding linear map $\bar{h} : \tilde{\otimes}_\pi^k E \rightarrow F$ the *linearisation* of $p \in \mathcal{P}({}^k E; F)$, and write $\bar{h} = \ell(p)$. Note that

$$\ell(p)(x \otimes x \otimes \dots \otimes x) = p(x) \quad \text{for all } x \in E.$$

Written in terms of $\ell(p)$, the Polarisation Formula becomes

$$\ell(p)(x_1 \otimes x_2 \otimes \dots \otimes x_k) = \frac{1}{2^k k!} \sum_{\epsilon_j = \pm 1} \epsilon_1 \epsilon_2 \dots \epsilon_k p(\epsilon_1 x_1 + \epsilon_2 x_2 + \dots + \epsilon_k x_k)$$

for all $(x_1, x_2, \dots, x_k) \in E^k$. This formula implies that the values of $\ell(p)$ throughout $\tilde{\otimes}_\pi^k E$ are determined by the values of the k -homogeneous polynomial map p on E .

A map $f : E \rightarrow F$ is said to be *holomorphic* (on E) if for each $a \in E$ there is a unique sequence of polynomial maps $\hat{d}^0 f(a), \hat{d}^1 f(a), \hat{d}^2 f(a), \dots$ such that the following conditions are satisfied:

- (1) each $\hat{d}^k f(a) \in \mathcal{P}({}^k E; F)$, and
- (2) the series $\sum_{k=0}^{\infty} \frac{1}{k!} \hat{d}^k f(a)(x - a)$ converges to $f(x)$ uniformly on some neighbourhood of a in E .

By (2) we mean that each $a \in E$ has a neighbourhood U satisfying

$$(\forall \varepsilon > 0)(\exists n \in \mathbb{N})(\forall x \in U) \quad m \geq n \Rightarrow \left\| f(x) - \sum_{k=0}^m \frac{1}{k!} \hat{d}^k f(a)(x - a) \right\| < \varepsilon.$$

Let $\mathcal{H}(E; F)$ denote the space of all holomorphic maps from E into F . Clearly for each k ,

$$\mathcal{P}({}^k E; F) \subseteq \mathcal{P}(E; F) \subseteq \mathcal{H}(E; F).$$

Asplund maps. Let us define a holomorphic map $f : E \rightarrow F$ to be an *Asplund map* if each $a \in E$ has a neighbourhood V whose separable subsets get mapped into absolutely convex $\sigma(F, F')$ -metrisable subsets of F . Any continuous linear map is holomorphic, so Theorem 4.4 says that a continuous linear map $T : E \rightarrow F$ is an Asplund map if and only if it factors through an Asplund space Z . In the case of a k -homogeneous polynomial map $p : E \rightarrow F$, we can extend that result as follows.

PROPOSITION 5.1. *For a k -homogeneous polynomial map $p : E \rightarrow F$, the following statements are equivalent:*

- (1) p is an Asplund map,
- (2) there is a neighbourhood V of 0 in E such that p maps the separable subsets of V into absolutely convex $\sigma(F, F')$ -metrisable subsets of F ,
- (3) $\ell(p)$ maps each bounded sequence in $\tilde{\otimes}_{\pi}^k E$ into an absolutely convex $\sigma(F, F')$ -metrisable subset of F ,
- (4) there exist an Asplund space Z , a k -homogeneous polynomial map $q : E \rightarrow Z$ and a continuous linear map $h : Z \rightarrow F$ such that $p = h \circ q$, and
- (5) p maps the bounded separable subsets of E into $\sigma(F, F')$ -metrisable absolutely convex subsets of F .

PROOF: The implication (1) \Rightarrow (2) is trivial.

(2) \Rightarrow (3): Let (z_n) be a bounded sequence in $\tilde{\otimes}_\pi^k E$, and suppose that V is the neighbourhood of 0 described in (3). Since the closed absolutely convex hull of $\chi_k(V^k)$ is a basic neighbourhood of 0 in $\tilde{\otimes}_\pi^k E$, it absorbs $\{z_n : n \in \mathbb{N}\}$. According to the Polarisation Formula, the image of $\chi_k(V^k)$ under the linear map $\ell(p) : \tilde{\otimes}_\pi^k E \rightarrow F$ is contained in the absolutely convex hull of $p(V)$. Thus the sequence $(\ell(p)(z_n))$ lies in a multiple of the closed absolutely convex hull of a countable subset of $p(V)$.

(3) \Rightarrow (4): According to Theorem 4.4, there exists an Asplund space Z together with continuous linear maps $g : \tilde{\otimes}_\pi^k E \rightarrow Z$ and $h : Z \rightarrow F$ satisfying $\ell(p) = h \circ g$. Thus

$$p = \ell(p) \circ \chi_k \circ \Delta_k = h \circ (g \circ \chi_k \circ \Delta_k),$$

where $(g \circ \chi_k \circ \Delta_k) \in \mathcal{P}({}^k E; F)$.

(4) \Rightarrow (5): Let A be a bounded separable subset of E . Put $U = \{x \in E : \|q(x)\| < 1\}$. Since U is a neighbourhood of 0 in E , there is a $\lambda > 0$ such that $A \subseteq \lambda U$. But then $\|q(x)\| < \lambda^k$ for all $x \in A$, so $q(A)$ is a bounded separable subset of Z . Since Z is an Asplund space, $q(A)$ is contained in an absolutely convex $\sigma(Z, Z')$ -metrisable subset of Z and so $p(A) = h \circ q(A)$ is contained in an absolutely convex $\sigma(F, F')$ -metrisable subset of F .

(5) \Rightarrow (1): For each $a \in E$, let $V = \{x \in E : \|x - a\| < 1\}$. The separable subsets of V are bounded in E , so according to (5) they are mapped into absolutely convex $\sigma(F, F')$ -metrisable subsets of F . Thus each $a \in E$ has a neighbourhood whose separable subsets get mapped into absolutely convex $\sigma(F, F')$ -metrisable subsets of F . ■

We now wish to show that a holomorphic map $f : E \rightarrow F$ is an Asplund map if and only if the k -homogeneous polynomial maps $\hat{d}^k f(a)$ are Asplund maps for each $a \in E$, and that this is the case if and only if it is true at 0. We shall make use of the following "infinite-dimensional" version of Cauchy's Integral Formula.

LEMMA 5.2. Let $f : E \rightarrow F$ be a holomorphic map, and suppose that $a \in E$, $y \in E$ and $\varphi \in F'$. For each $\rho > 0$,

$$\frac{1}{k!} \varphi \circ \hat{d}^k f(a)(y - a) = \frac{1}{2\pi i} \oint_{|\lambda|=\rho} \frac{\varphi \circ f((1 - \lambda)a + \lambda y)}{\lambda^{k+1}} d\lambda \quad \text{for all } k.$$

PROOF: It is possible to prove this Lemma directly (see [26, Prop. 2, pg 21]). However we prefer to derive it from the Cauchy Integral Formula of classical Complex Analysis. Define $h : \mathbb{C} \rightarrow \mathbb{C}$ by

$$h(\lambda) = \varphi \circ f((1 - \lambda)a + \lambda y) \quad \text{for all } \lambda \in \mathbb{C}.$$

There is an $r > 0$ such that the Taylor expansion $\sum_{k=0}^{\infty} \frac{1}{k!} \hat{d}^k f(a)(x - a)$ converges to $f(x)$ uniformly on $U = \{x \in E : \|x - a\| < r\}$. Put

$$D = \{\lambda \in \mathbb{C} : |\lambda| \|y - a\| < r\}.$$

For each $\lambda \in D$,

$$\begin{aligned} h(\lambda) &= \varphi \circ \sum_{k=0}^{\infty} \frac{1}{k!} \hat{d}^k f(a)(\lambda y - \lambda a) \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \varphi \circ \hat{d}^k f(a)(y - a). \end{aligned}$$

This last expression is a power series in λ . The function h is thus analytic on D , and

$$h^{(k)}(0) = \varphi \circ \hat{d}^k f(a)(y - a) \quad \text{for all } k.$$

By the classical Cauchy Integral Formula, for any $\rho > 0$,

$$\frac{h^{(k)}(0)}{k!} = \frac{1}{2\pi i} \oint_{|\lambda|=\rho} \frac{h(\lambda)}{\lambda^{k+1}} d\lambda \quad \text{for all } k.$$

Written in terms of f , this equation expresses the desired result. ■

The Lemma above implies that for each $f \in \mathcal{H}(E; F)$ and $a, y \in E$,

$$\frac{1}{k!} \hat{d}^k f(a)(y - a) = \frac{1}{2\pi i} \oint_{|\lambda|=\rho} \frac{f((1 - \lambda)a + \lambda y)}{\lambda^{k+1}} d\lambda \quad \text{for all } k.$$

(Because f is F -valued, the contour integral on the right must be taken in the sense of a Pettis integral.)

The next Lemma is based on a very useful result of Ryan's (see [31, 3.1]). Recall that a subset V of E is balanced if for each $x \in V$, $|\lambda| \leq 1 \implies \lambda x \in V$.

LEMMA 5.3. Let $f : E \rightarrow F$ be a holomorphic map, and suppose that V is a balanced subset of E . Then for each $a \in E$,

$$\frac{1}{k!} \hat{d}^k f(a)(V) \subseteq (f(a + V))^{\circ\circ} \quad \text{for all } k.$$

PROOF: Let $x \in V$, and suppose that $\varphi \in (f(a + V))^{\circ}$. Applying Lemma 5.2 with $\rho = 1$, we get

$$\begin{aligned} \frac{1}{k!} |\varphi(\hat{d}^k f(a)(x))| &\leq \frac{1}{2\pi} \oint_{|\lambda|=1} \left| \frac{\varphi \circ f((1-\lambda)a + \lambda(a+x))}{\lambda^{k+1}} \right| d\lambda \\ &\leq \frac{1}{2\pi} \cdot 2\pi \sup_{|\lambda|=1} \left| \frac{\varphi \circ f(a + \lambda x)}{\lambda^{k+1}} \right| \\ &\leq \sup_{y \in a+V} |\varphi \circ f(y)| \\ &\leq 1. \end{aligned}$$

Thus $\frac{1}{k!} \hat{d}^k f(a)(x) \in (f(a + V))^{\circ\circ}$ for all $x \in V$. ■

The next result appears in Nachbin's book [26, pg 26], but with a slightly different proof.

LEMMA 5.4. Let $f : E \rightarrow F$ be a holomorphic map, and suppose that $a \in E$. Then the Taylor expansion of f about the point a converges to $f(x)$ uniformly on a neighbourhood of each $y \in E$.

PROOF: From Lemma 5.2 we have that for each $\rho > 0$,

$$\begin{aligned} \left\| \frac{1}{k!} \hat{d}^k f(a)(x - a) \right\| &\leq \frac{1}{2\pi} \oint_{|\lambda|=\rho} \left\| \frac{1}{\lambda^{k+1}} f((1-\lambda)a + \lambda x) \right\| d\lambda \\ &\leq \frac{1}{2\pi} \cdot 2\pi \rho \cdot \frac{1}{\rho^{k+1}} \sup_{|\lambda|=\rho} \|f((1-\lambda)a + \lambda x)\| \\ &\leq \frac{1}{\rho^k} \sup_{|\lambda|=\rho} \|f((1-\lambda)a + \lambda x)\| \quad \text{for all } k. \end{aligned}$$

If $\rho > 1$, then for each $m \in \mathbb{N}$,

$$\begin{aligned} \left\| f(x) - \sum_{k=0}^m \frac{1}{k!} \hat{d}^k f(a)(x-a) \right\| &= \left\| \sum_{k=m+1}^{\infty} \frac{1}{k!} \hat{d}^k f(a)(x-a) \right\| \\ &\leq \sum_{k=m+1}^{\infty} \frac{1}{\rho^k} \cdot \sup_{|\lambda|=\rho} \|f((1-\lambda)a + \lambda x)\| \\ &\leq \frac{1}{\rho^m(\rho-1)} \cdot \sup_{|\lambda|=\rho} \|f((1-\lambda)a + \lambda x)\|. \end{aligned}$$

Since the map which takes each (x, λ) to $\|f((1-\lambda)a + \lambda x)\|$ is continuous on $E \times \mathbb{C}$, it is possible to find a $\rho > 1$ and a neighbourhood U of $y \in E$ such that

$$\sup_{x \in U} \sup_{|\lambda|=\rho} \|f((1-\lambda)a + \lambda x)\| < \infty.$$

Thus $\sum_{k=0}^{\infty} \frac{1}{k!} \hat{d}^k f(a)(x-a)$ converges to $f(x)$ uniformly on U . ■

THEOREM 5.5. *For a holomorphic map $f : E \rightarrow F$, the following statements are equivalent:*

- (1) f is an Asplund map,
- (2) there exists a neighbourhood U of 0 in E with the property that the image under f of each separable subset of U is contained in an absolutely convex $\sigma(F, F')$ -metrisable subset of F ,
- (3) for every $a \in E$, each $\hat{d}^k f(a)$ is an Asplund map, and
- (4) each $\hat{d}^k f(0)$ is an Asplund map.

PROOF: The implications (1) \Rightarrow (2) and (3) \Rightarrow (4) are trivial.

(1) \Rightarrow (3) and (2) \Rightarrow (4): For each $a \in E$, let V be a neighbourhood of 0 with the property that the image of each separable subset of $a + V$ is contained in a $\sigma(F, F')$ -metrisable subset of F . According to Lemma 5.3,

$$\frac{1}{k!} \hat{d}^k f(a)(V) \subseteq (f(a + V))^{\circ\circ} \quad \text{for all } k.$$

Thus $\hat{d}^k f(a)$ maps the separable subsets of V into $\sigma(F, F')$ -metrisable absolutely convex subsets of F . Since $\hat{d}^k f(a)$ is a k -homogeneous polynomial map, we can apply Proposition 5.1 to deduce that $\hat{d}^k f(a)$ is an Asplund map.

(4) \Rightarrow (1): According to Lemma 5.4, each $a \in E$ has a neighbourhood U such that $\sum_{k=0}^{\infty} \frac{1}{k!} \hat{d}^k f(0)(x)$ converges to $f(x)$ uniformly on U . Thus for each $m \in \mathbb{N}$ there is an n for which

$$\left\| f(x) - \sum_{k=0}^n \frac{1}{k!} \hat{d}^k f(0)(x) \right\| < 2^{-m} \quad \text{for all } x \in U.$$

Without any loss of generality, we can assume that U is bounded. We can now apply Lemma 4.3 to see that if a subset A of U has the property that $\sum_{k=0}^n \frac{1}{k!} \hat{d}^k f(0)(A)$ is $\sigma(F, F')$ -metrisable for each $n \in \mathbb{N}$, then $f(A)$ is itself $\sigma(F, F')$ -metrisable. ■

\mathcal{A}^d -holomorphic polynomial maps. The definition of an Asplund map given above is obviously similar to the definitions of compact, weakly compact and Rosenthal holomorphic maps in [1], [31] and [24]. The question arises as to whether all these definitions might be placed in a common framework. If so, it should be possible to give a unified proof of Proposition 5.1 and the corresponding results in the abovementioned papers.

An obvious approach is to make use of the notion of an operator ideal. Let \mathcal{A} be an operator ideal consisting of continuous linear maps between Banach spaces. For each pair of Banach spaces X, Y , put

$$\mathcal{A}(X; Y) = \mathcal{A} \cap \mathcal{L}(X; Y).$$

Again E and F will be a fixed pair of complex Banach spaces. We say that a map $f : E \rightarrow F$ has the \mathcal{A} -factorisation property on a subset U of E if there exists a Banach space Z together with a pair of maps $g \in \mathcal{H}(E; Z)$, $h \in \mathcal{A}(Z; F)$ such that

$$f(x) = h \circ g(x) \quad \text{for all } x \in U.$$

If f has the \mathcal{A} -factorisation property on a neighbourhood of each $a \in E$, then f is \mathcal{A} -holomorphic (on E).

This definition is motivated by the following special cases.

- (1) If \mathcal{A} is the ideal \mathcal{K} of all compact linear maps between Banach spaces, then the \mathcal{A} -holomorphic maps are precisely the compact holomorphic maps studied by Aron and Schottenloher.

- (2) Similarly, if \mathcal{A} is the ideal \mathcal{W} of weakly compact linear maps, or the ideal \mathcal{R} of Rosenthal linear maps, then we get the classes of weakly compact and Rosenthal holomorphic maps studied by Ryan and Lindström respectively.
- (3) If \mathcal{A} is the ideal of all continuous linear maps $T : X \rightarrow Y$ between Banach spaces that take bounded sequences in X into $\sigma(Y, Y')$ -metrisable absolutely convex subsets of Y , then the \mathcal{A} -holomorphic k -homogeneous polynomial maps are the Asplund k -homogeneous polynomial maps in the sense defined above.

We shall concentrate on \mathcal{A} -holomorphic k -homogeneous polynomial maps. The following result is useful. Recall that an operator ideal \mathcal{A} is said to be *surjective* if every continuous linear map $T : X \rightarrow Y$ between Banach spaces that satisfies $T \circ S \in \mathcal{A}$ for some continuous linear surjection $S : Z \rightarrow X$ (Z a Banach space) is itself in \mathcal{A} . It can be seen that every operator ideal satisfying a DFJP-type factorisation result is surjective, as is the ideal \mathcal{K} .

LEMMA 5.6. *Let \mathcal{A} be a surjective operator ideal consisting of continuous linear maps between Banach spaces. A k -homogeneous polynomial map $p : E \rightarrow F$ has the \mathcal{A} -factorisation property on a neighbourhood U of $\mathbf{0}$ in E if and only if there exist a Banach space \hat{X} , a continuous linear map $\hat{h} \in \mathcal{A}(\hat{X}; F)$ and a k -homogeneous polynomial map $\hat{g} \in \mathcal{P}(^k E; \hat{X})$ such that*

$$p(x) = \hat{h} \circ \hat{g}(x) \quad \text{for all } x \in U.$$

PROOF: Suppose that $p \in \mathcal{P}(^k E; F)$ has the \mathcal{A} -factorisation property on a neighbourhood U of $\mathbf{0}$ in E . This means that $p(x) = \hat{h} \circ g(x)$ for all $x \in U$, where $h \in \mathcal{A}(X; F)$ and $g \in \mathcal{H}(E; X)$ for some Banach space X . Let \hat{X} be the quotient space $X/\ker h$. Using the canonical surjection $\mathfrak{q} : X \rightarrow \hat{X}$, we can define $\hat{g} : E \rightarrow \hat{X}$ and $\hat{h} : \hat{X} \rightarrow F$ by $\hat{g} = \mathfrak{q} \circ g$ and $\hat{h} \circ \mathfrak{q} = h$. Because \mathcal{A} is surjective, it follows immediately that $\hat{h} \in \mathcal{A}(\hat{X}; F)$. We need therefore only check that $\hat{g} : E \rightarrow \hat{X}$ is a k -homogeneous polynomial map. To do this, it is sufficient to show that $\hat{g} = (\bar{g} \circ \chi_k) \circ \Delta_k$ for some continuous linear map $\bar{g} : \tilde{\otimes}_{\pi}^k E \rightarrow \hat{X}$. Let

$\ell(p)$ be the linearisation of p . We have that $p = \ell(p) \circ (\chi_k \circ \Delta_k)$. According to the Polarisation Formula, the range of $\ell(p)$ is contained in the linear span of the range of p , and this linear span is contained in the range of \hat{h} . It is therefore possible to define the desired map $\bar{g} : \tilde{\otimes}_\pi^k E \rightarrow \hat{X}$ by $\hat{h} \circ \bar{g} = \ell(p)$. ■

PROPOSITION 5.7. *Let \mathcal{A} be an surjective operator ideal consisting of continuous linear maps between Banach spaces. For each k -homogeneous polynomial map $p : E \rightarrow F$, the following statements are equivalent:*

- (1) p is \mathcal{A} -holomorphic on E ,
- (2) p has the \mathcal{A} -factorisation property on a neighbourhood of 0 in E ,
- (3) $\ell(p) \in \mathcal{A}(\tilde{\otimes}_\pi^k E; F)$, and
- (4) p has the \mathcal{A} -factorisation property on E .

PROOF: The implications (1) \Rightarrow (2), (3) \Rightarrow (4) and (4) \Rightarrow (1) are trivial.

(2) \Rightarrow (3): Suppose that p has the \mathcal{A} -factorisation property on a neighbourhood U of 0 in E . According to the previous Lemma, there is a Banach space X together with maps $g : E \rightarrow X$ and $h : X \rightarrow F$ such that $g \in \mathcal{H}(E; X)$, $h \in \mathcal{A}(X; F)$ and $p(x) = h \circ g(x)$ for all $x \in U$. Because p is a k -homogeneous polynomial map, so is g . Thus $g = \ell(g) \circ \chi_k \circ \Delta_k$. By the Polarisation Formula,

$$\ell(p)(x_1 \otimes x_2 \otimes \cdots \otimes x_k) = h \circ \ell(g)(x_1 \otimes x_2 \otimes \cdots \otimes x_k) \quad \text{for all } x_1, x_2, \dots, x_k \in U.$$

Because U is an absorbent subset of E and each of the maps $\ell(p)$, h and $\ell(g)$ is linear, we can deduce that

$$\ell(p) = h \circ \ell(g).$$

Since $h \in \mathcal{A}(X; F)$, the map $h \circ \ell(g)$ belongs to $\mathcal{A}(\tilde{\otimes}_\pi^k E; F)$. ■

We shall denote the space of complex-valued k -homogeneous polynomial maps on E by $\mathcal{P}(^k E)$. There is a norm on $\mathcal{P}(^k E)$ given by

$$\|q\| = \sup\{|q(x)| : x \in E, \|x\| \leq 1\} \quad \text{for all } q \in \mathcal{P}(^k E).$$

Every k -homogeneous polynomial map $p : E \rightarrow F$ has a transpose $p^\top : F' \rightarrow \mathcal{P}(^k E)$ defined by

$$p^\top(\phi) = \phi \circ p \quad \text{for all } \phi \in F'.$$

This transpose is always linear and norm-continuous.

Let \mathcal{A} be an operator ideal consisting of continuous linear maps between Banach spaces. The *dual ideal* \mathcal{A}^d is defined to consist of those continuous linear maps between Banach spaces whose adjoints belong to \mathcal{A} . Thus for each pair of Banach spaces X, Y ,

$$\mathcal{A}^d(X; Y) = \{T \in \mathcal{L}(X; Y) : T' \in \mathcal{A}(Y'; X')\}.$$

It follows from Schauder's Theorem [19, 17.1.3] that $\mathcal{K}^d = \mathcal{K}$, and from Gantmacher's Theorem [19, 17.2.5] that $\mathcal{W}^d = \mathcal{W}$. The situation for the ideals of Rosenthal and Asplund continuous linear maps is a little more complicated, as these ideals do not satisfy $\mathcal{A}^d = \mathcal{A}$.

Recall that an operator ideal is said to be *injective* if for each pair of Banach spaces X, Y , every continuous linear map $T : X \rightarrow Y$ which satisfies $J \circ T \in \mathcal{A}(X; Z)$ for some continuous linear injection $J : Y \rightarrow Z$ (where Z is a Banach space) belongs to $\mathcal{A}(X; Y)$.

PROPOSITION 5.8. *Let \mathcal{A} be operator ideal consisting of continuous linear maps between Banach spaces. Provided that \mathcal{A} is both injective and surjective, a k -homogeneous polynomial map $p : E \rightarrow F$ is \mathcal{A}^d -holomorphic if and only if its transpose p^\top belongs to $\mathcal{A}(F'; \mathcal{P}(^k E))$.*

PROOF: First note that a polynomial map p is \mathcal{A}^d -holomorphic if and only if $\ell(p)' \in \mathcal{A}(F'; (\tilde{\otimes}_\pi^k E)')$. Since $p = \ell(p) \circ \chi_k \circ \Delta_k$, $p^\top = (\chi_k \circ \Delta_k)^\top \circ \ell(p)'$. Thus whenever $\ell(p)' \in \mathcal{A}(F'; (\tilde{\otimes}_\pi^k E)')$, $p^\top \in \mathcal{A}(F'; \mathcal{P}(^k E))$. Conversely, if we know that $p^\top \in \mathcal{A}(F'; \mathcal{P}(^k E))$, then by the injectivity of $(\chi_k \circ \Delta_k)^\top$, $\ell(p)' \in \mathcal{A}(F'; (\tilde{\otimes}_\pi^k E)')$. ■

The above Proposition generalises [1, 3.2] and [31, 2.1].

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